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A FLEXIBLE HYPERSONIC VEHICLE MODEL DEVELOPED WITH PISTON THEORY (PREPRINT)

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14. ABSTRACT

For high Mach number flows, M > 4, piston theory has been used to calculate the pressures on the surfaces of a vehicle. In a two-dimensional inviscid flow, a perpendicular column of fluid stays intact as it passes over a solid surface. Thus, the pressure at the surface can be calculated assuming the surface were a piston moving into a column of fluid. In this work, first-order piston theory is used to calculate the forces, moments, and stability derivatives for longitudinal motion of a hypersonic vehicle. The advantage of piston theory over other techniques, such as Prandtl-Meyer flow, oblique shock, or Newtonian impact theory, is that unsteady aerodynamic effects can be included in the model. The unsteady effects, considered in this work, include perturbations in the linear velocities and angular rates, due to rigid body motion. A flexible model is developed using Euler-Bernoulli beam theory and the unsteady effects due to aeroelastics are captured in the model.

15. SUBJECT TERMS

Piston Theory, Unsteady Aerodynamics, Aeroelastic Model

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A Flexible Hypersonic Vehicle Model Developed With Piston Theory

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I. Abstract

For high Mach number flows, $M \geq 4$, piston theory has been used to calculate the pressures on the surfaces of a vehicle. In a two-dimensional inviscid flow, a perpendicular column of fluid stays intact as it passes over a solid surface. Thus, the pressure at the surface can be calculated assuming the surface were a piston moving into a column of fluid. In this work, first-order piston theory is used to calculate the forces, moments, and stability derivatives for longitudinal motion of a hypersonic vehicle. Piston theory predicts a relationship between the local pressure on a surface and the normal component of fluid velocity produced by the surface's motion. The advantage of piston theory over other techniques, such as Prandtl-Meyer flow, oblique shock, or Newtonian impact theory, is that unsteady aerodynamic effects can be included in the model. The unsteady effects, considered in this work, include perturbations in the linear velocities and angular rates, due to rigid body motion. This provides a more accurate model that agrees more closely with models derived using computational fluid dynamics or those derived by solving Euler equations. Additionally, piston theory yields an analytical model for the longitudinal motion of the vehicle, thus allowing design trade studies to be performed while still providing insight into the physics of the problem.

II. Introduction

In the 1980's, the National Aerospace Plane (NASP) program commenced, with its goal being a feasibility study for a single-stage to orbit (SSTO) vehicle, which was reusable and could take off and land horizontally. The NASP was to be powered by a supersonic combustion ramjet (scramjet) engine. Although this program was cancelled in the 1990's, a great deal of knowledge was gained and it spawned future programs, including the hypersonic systems technology program (HySTP), initiated in late 1994, and the NASA X-43A. The HySTP's goal was to transfer the accomplishments of the NASP program to a technology demonstration program. This program was cancelled in early 1995. The NASA X-43A set new world speed records in 2004, reaching Mach 6.8 and Mach 9.6 on two separate occasions with a scramjet engine. These flights were the culmination of NASA's Hyper-X program, with the objective being to explore alternatives to rocket power for space access vehicles.

With renewed interest in space operations worldwide, there is an interest in hypersonic aerodynamics research. The scramjet engine will likely play a major role in future hypersonic vehicles. Unlike a conventional turbojet engine, a scramjet engine does not use high speed turbomachinery to compress the air before it reaches the combustor. Instead, it relies upon the rise in pressure across oblique shock waves located in front of the inlet. Furthermore, the flow through the entire engine is supersonic in contrast to a ramjet, where the flow speeds are subsonic through the combustor. On configurations like the NASP and X-43A, the underside of the airframe must function as the air inlet mechanism and the exhaust nozzle. Therefore, integration of the airframe and engine are critical to success of a scramjet powered vehicle.

Scramjets could be used as part of a multi-stage launch vehicle that would include multiple propulsion systems to perform a mission. The factor driving research towards scramjets and away from rockets is cost;

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scramjets would substantially lower costs because it is an airbreathing engine. Airbreathing engines don't require oxidizer to be carried by the vehicle, hence increasing the payload and reducing the quantity of fuel carried.

Unsteady aerodynamics is another key technology in the development and optimization of future hypersonic vehicles. The combined effects of a slender flexible vehicle travelling at high speeds and subjected to large forces may lead to significant unsteady aerodynamic effects. Hence, understanding the concepts and consequences of time-dependent aerodynamic flows is critical to the success of this type of vehicle.

Piston theory is a technique which has been used for years to determine the pressure distributions on an airfoil/vehicle, when the Mach number is sufficiently high. Lighthill[1] discussed the application of piston theory on oscillating airfoils some 50 years ago. Ashley and Zartarian[2] discuss piston theory while providing a number of examples of the application of piston theory to specific problems. More recently, Tarpley[3] discussed the computation of stability derivatives for a caret-wing waverider using piston theory, which requires the analysis of unsteady flow over the vehicle.[3] Piston theory allows the inclusion of unsteady aerodynamic effects in the model and a closed form solution can be found for these unsteady effects.

In this work, piston theory is applied to a hypothetical 2-dimensional hypersonic vehicle powered with a scramjet. Section III describes the vehicle analyzed in this work, while the steady forces on the vehicle's surfaces are calculated in Section IV. The afterbody effects are included in Section V, where the pressure distribution and force due to exhaust plume are evaluated. The control surface, which is a single elevator, is examined in Section VI, analysis of the flow regions is performed in Section VII, total body forces and moments are derived in Section VIII, while stability derivatives are calculated in Section IX. The engine model is developed in Section X and some results from an open-loop simulation are provided in Section XIII.

III. HSV Model

Figure 1 shows the hypersonic vehicle considered in this work.[4] The vehicle consists of 4 surfaces:

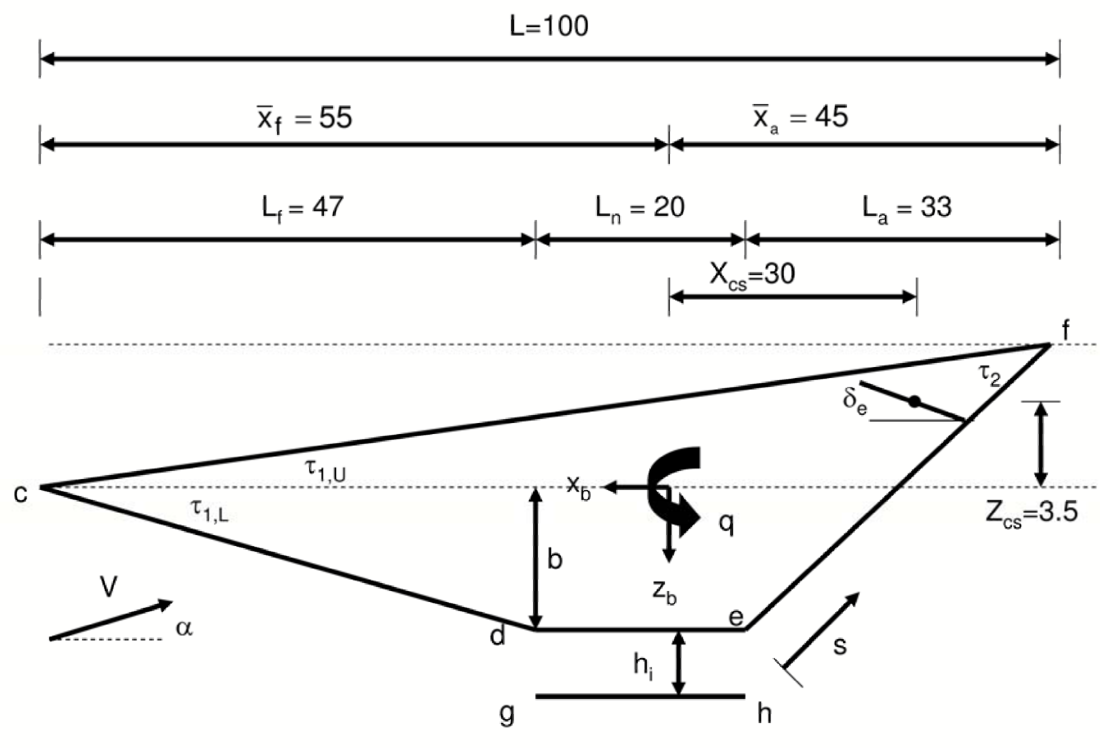

Figure 1. Hypersonic Vehicle.

an upper surface (defined by points cf) and three lower surfaces (defined by points cd, gh, and ef). All pertinent lengths and dimensions are in units of feet and degrees, respectively. The total length of the vehicle is $L = 100ft$ and the notation for lengths is L_f = length of the forebody, L_n = length of the engine nacelle, L_a = length of the aftbody, L_e is the length of the elevator, $\bar{x}_f$ is the distance from the C.G. to the front of the vehicle, $\bar{x}_a$ is the distance from the C.G. to the rear of the vehicle, x_{cs} and z_{cs} are the distances from the C.G. to the midpoint of the elevator in the x and z directions, respectively, and h_i is the engine height. The vehicle lengths are

$$\begin{aligned} L &= 100\ ft \\ L_f &= 47\ ft \\ L_a &= 33\ ft \\ L_n &= 20\ ft \\ L_e &= 17\ ft \\ \bar{x}_f &= 55\ ft \\ \bar{x}_a &= 45\ ft \\ x_{cs} &= 30\ ft \\ z_{cs} &= 3.5\ ft \\ h_i &= 3.5\ ft \end{aligned} \tag{1}$$

The vehicle angles are

$$\begin{aligned} \tau_{1,U} &= 3^\circ \\ \tau_{1,L} &= 6^\circ \\ \tau_2 &= 14.41^\circ \end{aligned} \tag{2}$$

Additionally, the vehicle mass and moment of inertia are

$$\begin{aligned} Mass &= 300\ slug \\ J_{yy} &= 500,000\ slug - ft^2 \end{aligned} \tag{3}$$

and the mean aerodynamic chord ($\bar{c}$) and planform area (S) are defined as

$$\begin{aligned} \bar{c} &= L \\ S &= L^2 \end{aligned} \tag{4}$$

The goal is to apply piston theory to this vehicle to determine the pressure distribution on the surfaces of the vehicle, which, in turn, can be used to evaluate the forces and moments. The pressure on the face of a piston moving into a column of perfect gas is[2]

$$\frac{P}{P_\infty} = \left(1 + \frac{\gamma - 1}{2}\frac{V_n}{a_\infty}\right)^{\frac{2\gamma}{\gamma-1}} \tag{5}$$

where the subscript "∞" refers to the steady flow conditions past the surface, V_n is the velocity of the surface normal to the steady flow, a_∞ is the freestream speed of sound, and P is the surface pressure. Taking the binomial expansion of Eq. 5 produces

$$\frac{P}{P_\infty} = 1 + \frac{2\gamma}{\gamma - 1}\frac{\gamma - 1}{2}\frac{V_n}{a_\infty} = 1 + \frac{\gamma V_n}{a_\infty} \tag{6}$$

Multiplying through by P_∞ and using the perfect gas law ($P = \rho RT$) and the definition of the speed of sound ($a^2 = \gamma RT$) yields the basic result from first-order linear piston theory

$$P = P_\infty + \rho_\infty a_\infty V_n \tag{7}$$

where γ is the ratio of specific heats and R is the gas constant. The infinitesimal force due to the pressure is

$$d\mathbf{F} = -PdA\mathbf{n} \tag{8}$$

where dA is a surface element and $\mathbf{n}$ is the outward pointing normal. Substituting Eq. 7 into Eq. 8 yields

$$d\mathbf{F} = \left(-P_\infty - \rho_\infty a_\infty V_n\right) dA\mathbf{n} \tag{9}$$

The normal velocity can be computed by taking the dot product of the flow velocity over a surface and the outward pointing normal for that surface. Hence, Eq. 9 becomes

$$d\mathbf{F} = \left(-P_\infty - \rho_\infty a_\infty \left[\mathbf{V}\cdot\mathbf{n}\right]\right) dA\mathbf{n} \tag{10}$$

Equation 10 is the basic result upon which this work is based. From this equation, it is seen that in order to compute the forces acting on a surface, one must determine the properties of the flow past the surface (properties behind a shock, expansion fan, or freestream), the velocity of the surface relative to the airstream, $\mathbf{V}$, the outward pointing surface normal, $\mathbf{n}$, and the surface element, dA. The work that follows will develop these quantities for the upper surface and the lower surfaces.

IV. Vehicle Surface Pressure Distributions and Forces

To compute the forces, moments, and stability derivatives, consider small perturbations, from a steady flight condition at M_∞, in the velocities u and w and the rate q. On the upper surface, the surface is modelled as a piston moving into a column of fluid that has the properties of the flow behind an oblique shock wave, an expansion fan, or freestream (flow properties are determined by the angle of attack). Likewise, on the lower surface, the surface is modelled as a piston moving into a column of fluid that has the properties of the fluid behind the oblique shock. An oblique shock is required on the lower surface for proper engine operation. Figure 2 shows the regions of interest.

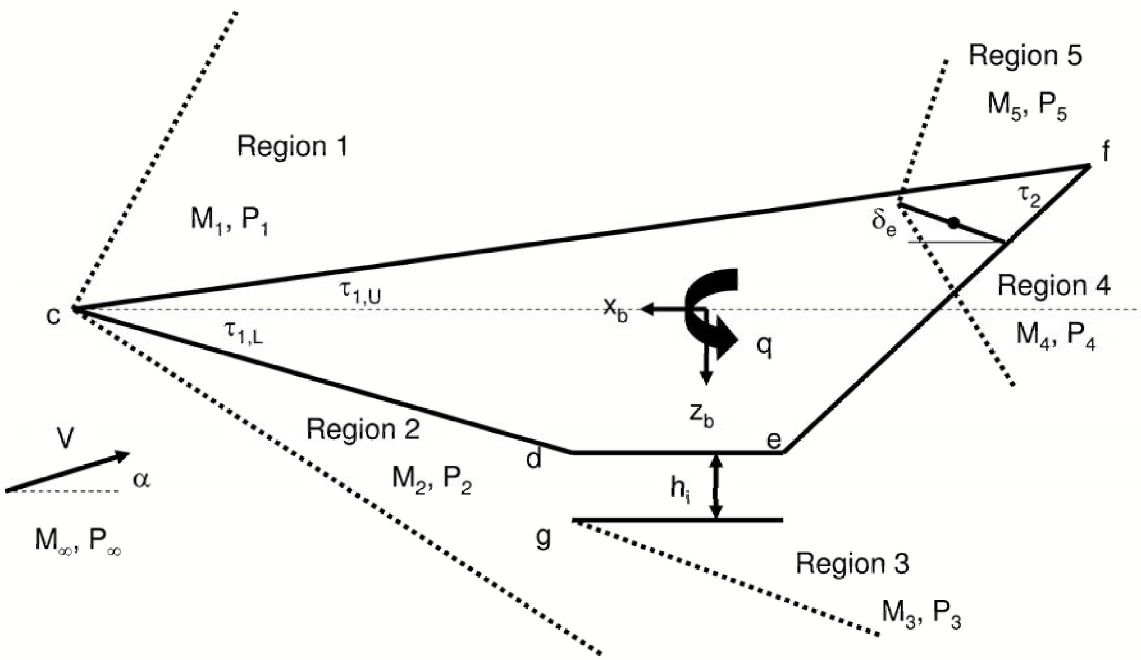

Figure 2. Hypersonic Vehicle, Oblique Shocks, and Pressure Regions.

To begin the development, first consider the upper surface. The velocity of a point on the upper surface due to the velocity and rate perturbations is

$$\mathbf{V}_{cf} = \left(V_1 \cos\tau_{1,U} + u\right)\hat{\imath} + \left(V_1 \sin\tau_{1,U} + w\right)\hat{k} + \boldsymbol{\omega}\times\mathbf{r}_{cf} \tag{11}$$

where $\hat{\imath}, \hat{k}$ are unit vectors in the x and z body axes, respectively, $\boldsymbol{\omega}$ is the angular rate vector, α is the angle of attack, and V_1 is the velocity of the flow on the upper surface (see region 1 in Fig. 2). For longitudinal motion only, $\boldsymbol{\omega} = q\hat{\jmath}$ where $\hat{\jmath}$ is a unit vector in the y body axis direction. In Eq. 11, $\mathbf{r}_{cf}$ is the position vector of a point on the upper surface given by

$$\begin{gathered}\mathbf{r}_{cf} = r_{cf_x}\hat{\imath} + +r_{cf_z}\hat{k} = x\hat{\imath} + \tan\tau_{1,U}\left(x - \bar{x}_f\right)\hat{k} \\ -\bar{x}_a \leq x \leq \bar{x}_f\end{gathered} \tag{12}$$

According to Eq. 8, a normal vector to the upper surface is also needed. The upper surface outward pointing normal vector is

$$\mathbf{n}_{cf} = \sin\tau_{1,U}\hat{\imath} - \cos\tau_{1,U}\hat{k} \tag{13}$$

For the lower surface defined by the points c and d in Figure 1, we use the velocity of the flow after the oblique shock to obtain

$$\mathbf{V}_{cd} = (V_2\cos\tau_{1,L} + u)\,\hat{\imath} + (-V_2\sin\tau_{1,L} + w)\,\hat{k} + \boldsymbol{\omega}\times\mathbf{r}_{cd} \tag{14}$$

while for the surface defined by points g and h

$$\mathbf{V}_{gh} = (V_3 + u)\,\hat{\imath} + w\hat{k} + \boldsymbol{\omega}\times\mathbf{r}_{gh} \tag{15}$$

where $\mathbf{r}_{cd}$ and $\mathbf{r}_{gh}$ are position vectors of a point on the lower surface given by

$$\begin{gathered}\mathbf{r}_{cd} = r_{cd_x}\hat{\imath} + r_{cd_z}\hat{k} = x\hat{\imath} - \tan\tau_{1,L}\,(x - \bar{x}_f)\,\hat{k} \\ \bar{x}_f - L_f \le x \le \bar{x}_f\end{gathered} \tag{16}$$

$$\begin{gathered}\mathbf{r}_{gh} = r_{gh_x}\hat{\imath} + r_{gh_z}\hat{k} = x\hat{\imath} + (L_f\tan\tau_{1,L} + h_i)\,\hat{k} \\ (\bar{x}_f - L_f) - L_n \le x \le \bar{x}_f - L_f\end{gathered} \tag{17}$$

The normal vectors for the lower surfaces are

$$\begin{gathered}\mathbf{n}_{cd} = \sin\tau_{1,L}\hat{\imath} + \cos\tau_{1,L}\hat{k} \\ \mathbf{n}_{gh} = 1\hat{k}\end{gathered} \tag{18}$$

Performing the cross products required by Eqs. 11, 14, and 15 gives

$$\boldsymbol{\omega}\times\mathbf{r}_{cf} = q\tan\tau_{1,U}\,(x - \bar{x}_f)\,\hat{\imath} - qx\hat{k} \tag{19}$$

$$\boldsymbol{\omega}\times\mathbf{r}_{cd} = -q\tan\tau_{1,L}\,(x - \bar{x}_f)\,\hat{\imath} - qx\hat{k} \tag{20}$$

$$\boldsymbol{\omega}\times\mathbf{r}_{gh} = q\,(L_f\tan\tau_{1,L} + h_i)\,\hat{\imath} - qx\hat{k} \tag{21}$$

According to Eq. 10, the pressures on the surfaces of interest are

$$\begin{gathered}P_{cf} = P_1 + \rho_1 a_1\,(\mathbf{V}_{cf}\cdot\mathbf{n}_{cf}) \\ P_{cd} = P_2 + \rho_2 a_2\,(\mathbf{V}_{cd}\cdot\mathbf{n}_{cd}) \\ P_{gh} = P_3 + \rho_3 a_3\,(\mathbf{V}_{gh}\cdot\mathbf{n}_{gh})\end{gathered} \tag{22}$$

Substituting the results of Eq. 22 into Eq. 8 gives

$$\begin{gathered}d\mathbf{F}_{cf} = \{-P_1 - \rho_1 a_1\,(\mathbf{V}_{cf}\cdot\mathbf{n}_{cf})\}\,dA_{cf}\mathbf{n}_{cf} \\ d\mathbf{F}_{cd} = \{-P_2 - \rho_2 a_2\,(\mathbf{V}_{cd}\cdot\mathbf{n}_{cd})\}\,dA_{cd}\mathbf{n}_{cd} \\ d\mathbf{F}_{gh} = \{-P_3 - \rho_3 a_3\,(\mathbf{V}_{gh}\cdot\mathbf{n}_{gh})\}\,dA_{gh}\mathbf{n}_{gh}\end{gathered} \tag{23}$$

Using Eqs. 11, 14, and 15 and the appropriate normal vectors (Eqs. 13 and 18), the dot products in Eq. 23 become

$$\begin{gathered}\mathbf{V}_{cf}\cdot\mathbf{n}_{cf} = [u + q\tan\tau_{1,U}\,(x - \bar{x}_f)]\sin\tau_{1,U} - [w - qx]\cos\tau_{1,U} \\ \mathbf{V}_{cd}\cdot\mathbf{n}_{cd} = [u - q\tan\tau_{1,L}\,(x - \bar{x}_f)]\sin\tau_{1,L} + [w - qx]\cos\tau_{1,L} \\ \mathbf{V}_{gh}\cdot\mathbf{n}_{gh} = w - qx\end{gathered} \tag{24}$$

Note that the steady terms cancel as a result of taking the dot product. Using Eq. 24 in Eq. 23 yields

$$\begin{gathered}d\mathbf{F}_{cf} = (-P_1 - \rho_1 a_1\,\{[u + q\tan\tau_{1,U}\,(x - \bar{x}_f)]\sin\tau_{1,U} - [w - qx]\cos\tau_{1,U}\})\,dA_{cf}\mathbf{n}_{cf} \\ d\mathbf{F}_{cd} = (-P_2 - \rho_2 a_2\,\{[u - q\tan\tau_{1,L}\,(x - \bar{x}_f)]\sin\tau_{1,L} + [w - qx]\cos\tau_{1,L}\})\,dA_{cd}\mathbf{n}_{cd} \\ d\mathbf{F}_{gh} = (-P_3 - \rho_3 a_3\,\{w - qx\})\,dA_{gh}\mathbf{n}_{gh}\end{gathered} \tag{25}$$

The next step is to determine the upper and lower surface elements. Note that the vehicle model is 2-dimensional with unit depth into the page. Hence, the upper surface element, dA_{cf} can be written as

$$dA_{cf} = dL_{cf}(1) \tag{26}$$

where dL_{cf} defines a length of interest on the upper surface and the multiplying factor of 1 is due to the vehicle's unit depth. The surface element can be written as

$$dA_{cf} = \sqrt{dx^2 + dz^2}(1) \tag{27}$$

From Eq. 12,

$$z = \tan\tau_{1,U}\,(x - \bar{x}_f) \Rightarrow dz = \tan\tau_{1,U}dx \tag{28}$$

Using Eq. 28 in Eq. 27 yields

$$dA_{cf} = \sqrt{dx^2 + \tan^2\tau_{1,U}dx^2}(1) = dx\sqrt{1 + \tan^2\tau_{1,U}}(1) = dx\sec\tau_{1,U}(1) = \sec\tau_{1,U}dx \tag{29}$$

Similarly, the surface elements for the lower surfaces become

$$dA_{cd} = \sec\tau_{1,L}dx \tag{30}$$

$$dA_{gh} = dx \tag{31}$$

Now, the incremental force on the upper surface (see first entry in Eq. 25) becomes

$$d\mathbf{F}_{cf} = (-P_1 - \rho_1 a_1\left\{[u + q\tan\tau_{1,U}\,(x - \bar{x}_f)]\sin\tau_{1,U} - [w - qx]\cos\tau_{1,U}\right\})\,\mathbf{n}_{cf}\sec\tau_{1,U}dx \tag{32}$$

Using similar analysis, the incremental forces on the lower surfaces become

$$d\mathbf{F}_{cd} = (-P_2 - \rho_2 a_2\left\{[u - q\tan\tau_{1,L}\,(x - \bar{x}_f)]\sin\tau_{1,L} + [w - qx]\cos\tau_{1,L}\right\})\,\mathbf{n}_{cd}\sec\tau_{1,L}dx \tag{33}$$

$$d\mathbf{F}_{gh} = (-P_3 - \rho_3 a_3\left\{w - qx\right\})\,\mathbf{n}_{gh}dx \tag{34}$$

It will be useful to separate the steady forces from the unsteady forces at this point. In Section IX, we will model the unsteady effects using a stability derivative approach. To compute the steady forces, the components of the incremental forces related to the steady flow are integrated over the surface of the vehicle. For the upper surface, this becomes

$$\mathbf{F}_{cf} = \int_{-\bar{x}_a}^{\bar{x}_f} (d\mathbf{F}_{cf})_{steady} = \int_{-\bar{x}_a}^{\bar{x}_f} -P_1\left[\sin\tau_{1,U}\hat{\imath} - \cos\tau_{1,U}\hat{k}\right]\sec\tau_{1,U}dx \tag{35}$$

while, for the lower surfaces, the forces are

$$\mathbf{F}_{cd} = \int_{\bar{x}_f - L_f}^{\bar{x}_f} (d\mathbf{F}_{cd})_{steady} = \int_{\bar{x}_f - L_f}^{\bar{x}_f} -P_2\left[\sin\tau_{1,L}\hat{\imath} + \cos\tau_{1,L}\hat{k}\right]\sec\tau_{1,L}dx \tag{36}$$

$$\mathbf{F}_{gh} = \int_{(\bar{x}_f - L_f) - L_n}^{\bar{x}_f - L_f} (d\mathbf{F}_{gh})_{steady} = \int_{(\bar{x}_f - L_f) - L_n}^{\bar{x}_f - L_f} -P_3\hat{k}dx \tag{37}$$

For the consideration of aeroelastic effects in future work, the upper surface force is separated into forebody and afterbody components. Thus, Eq. 35 becomes

$$\begin{aligned}\mathbf{F}_{cf} &= \int_{-\bar{x}_a}^{0} -P_1\left[\sin\tau_{1,U}\hat{\imath} - \cos\tau_{1,U}\hat{k}\right]\sec\tau_{1,U}dx + \int_{0}^{\bar{x}_f} -P_1\left[\sin\tau_{1,U}\hat{\imath} - \cos\tau_{1,U}\hat{k}\right]\sec\tau_{1,U}dx \\ &= \mathbf{F}_{cf_a} + \mathbf{F}_{cf_f}\end{aligned} \tag{38}$$

Performing the integrations yields

$$\begin{aligned}\mathbf{F}_{cf_a} &= -P_1\bar{x}_a\sec\tau_{1,U}\left[\sin\tau_{1,U}\hat{\imath} - \cos\tau_{1,U}\hat{k}\right] = X_{cf_a}\hat{\imath} + Z_{cf_a}\hat{k} \\ \mathbf{F}_{cf_f} &= -P_1\bar{x}_f\sec\tau_{1,U}\left[\sin\tau_{1,U}\hat{\imath} - \cos\tau_{1,U}\hat{k}\right] = X_{cf_f}\hat{\imath} + Z_{cf_f}\hat{k}\end{aligned} \tag{39}$$

where X_{cf_a}, Z_{cf_a} are the components of the aftbody upper surface force in the x and z directions, respectively, and X_{cf_f}, Z_{cf_f} are the components of the forebody upper surface force in the x and z directions. For the lower surfaces, integration of Eqs. 36 and 37 produce

$$\begin{aligned}\mathbf{F}_{cd} &= -P_2 L_f \sec\tau_{1,L}\left[\sin\tau_{1,L}\hat{\imath} + \cos\tau_{1,L}\hat{k}\right] = X_{cd}\hat{\imath} + Z_{cd}\hat{k} \\ \mathbf{F}_{gh} &= -P_3 L_n \hat{k} = X_{gh}\hat{\imath} + Z_{gh}\hat{k}\end{aligned} \tag{40}$$

where X_* and Z_* are the components of the lower surface forces. Equations 39 and 40 give the rigid body steady forces on this vehicle due to pressure distributions on the upper and lower surfaces.

V. Afterbody

The flow on the afterbody of the vehicle is bounded by the vehicle surface (surface ef in Fig. 1) and a shear layer between the freestream atmosphere and the exhaust gas of the engine. Hence, external nozzle analysis must be performed to determine the pressures on surface ef. According to Chavez,[5] the pressure distribution along the nozzle surface can be approximated by

$$P_{ef}(s) \approx \frac{P_e}{1 + \frac{s}{\frac{L_a}{\cos\left(\tau_{1,U}+\tau_2\right)}}\left(\frac{P_e}{P_\infty} - 1\right)} \tag{41}$$

where P_{ef} is the pressure on the afterbody, P_e is the pressure at the engine exit, P_∞ is the freestream pressure, and s is the distance from the lower apex (point e) to the point of interest along the vehicle's afterbody surface (see Fig. 1). The force produced by the external nozzle can be calculated by integrating Eq. 41 over the rear ramp of the vehicle:

$$F_{ef} = \int_0^{\frac{L_a}{\cos\left(\tau_{1,U}+\tau_2\right)}} \frac{P_e}{1 + \frac{s}{\frac{L_a}{\cos\left(\tau_{1,U}+\tau_2\right)}}\left(\frac{P_e}{P_\infty} - 1\right)} ds \tag{42}$$

Performing the integration and simplifying yields

$$F_{ef} = \frac{L_a P_e P_\infty}{\cos\left(\tau_{1,U}+\tau_2\right)\left(P_e - P_\infty\right)} \ln\frac{P_e}{P_\infty} \tag{43}$$

Equation 43 provides the magnitude of the force due to the external nozzle. Its direction is perpendicular to the rear ramp. Hence, the vector force due to the external nozzle is

$$\mathbf{F}_{ef} = \frac{L_a P_e P_\infty}{\cos\left(\tau_{1,U}+\tau_2\right)\left(P_e - P_\infty\right)} \ln\frac{P_e}{P_\infty}\left[\sin\left(\tau_{1,U}+\tau_2\right)\hat{\imath} - \cos\left(\tau_{1,U}+\tau_2\right)\hat{k}\right] = X_{ef}\hat{\imath} + Z_{ef}\hat{k} \tag{44}$$

where X_{ef} and Z_{ef} are the axial and normal force components of the external nozzle force.

For use in calculating stability derivatives, it is necessary to determine the force on the rear ramp due to perturbations in the velocities u and w and the rate q. The differential force on the rear ramp is

$$d\mathbf{F}_{ef} = \left(-P_{ef} - \rho_{ef} a_{ef}\left[\mathbf{V}_{ef} \cdot \mathbf{n}_{ef}\right]\right) dA_{ef}\mathbf{n}_{ef} \tag{45}$$

where P_{ef} is the afterbody pressure distribution given by Eq. 41, ρ_{ef}, a_{ef} are the density and speed of sound on the rear ramp, $\mathbf{V}_{ef}$ is the flow velocity on lower surface ef, $\mathbf{n}_{ef}$ is the normal vector to lower surface ef, and dA_{ef} is the surface element for surface ef. The position vector, normal vector, and surface element for this surface are given by:

$$\begin{gathered}\mathbf{r}_{ef} = x\hat{\imath} + \left[\tan\left(\tau_{1,U}+\tau_2\right)\left(x + \bar{x}_a\right) - L\tan\tau_{1,U}\right]\hat{k} = x\hat{\imath} + r_{ef_z}\hat{k} \\ -\bar{x}_a \le x \le L_a - \bar{x}_a\end{gathered} \tag{46}$$

$$\mathbf{n}_{ef} = -\sin\left(\tau_{1,U}+\tau_2\right)\hat{\imath} + \cos\left(\tau_{1,U}+\tau_2\right)\hat{k} \tag{47}$$

$$dA_{ef} = \sec\left(\tau_{1,U}+\tau_2\right)dx \tag{48}$$

Using Eqs. 46-48 in Eq. 45 yields

$$d\mathbf{F}_{ef} = \left(-P_{ef} - \rho_{ef} a_{ef}\left\{-\left(u + q r_{ef_z}\right)\sin\left(\tau_{1,U}+\tau_2\right) + \left(w - qx\right)\cos\left(\tau_{1,U}+\tau_2\right)\right\}\right)\mathbf{n}_{ef} dA_{ef} \tag{49}$$

Integration of the steady component of Eq. 49 yields the same result as in Eq. 44. The unsteady components will be utilized later in this work for the computation of stability derivatives.

VI. Control Surfaces

The control surface is an elevator located near the tail of the vehicle as shown in Fig. 1. The elevator is modelled as a flat plate hinged at its midpoint so the entire surface deflects. The length of the elevator is $L_e = 17ft$. Positive δ_e is defined as trailing edge down. Once again, the velocity of the flow on both sides of the elevator must be determined. Proceeding in a manner similar to that which has already been done, it is found that

$$\mathbf{V}_{e_L} = (V_4 \cos\delta_e + u)\,\hat{\imath} + (-V_4 \sin\delta_e + w)\,\hat{k} + \boldsymbol{\omega}\times\mathbf{r}_e \tag{50}$$

and

$$\mathbf{V}_{e_U} = (V_5 \cos\delta_e + u)\,\hat{\imath} + (-V_5 \sin\delta_e + w)\,\hat{k} + \boldsymbol{\omega}\times\mathbf{r}_e \tag{51}$$

where V_{e_L} is the flow velocity on the underside of the elevator, V_{e_U} is the flow velocity on the upper surface of the elevator, $\boldsymbol{\omega} = q\hat{\jmath}$, $\mathbf{r}_e$ is a position vector from the vehicle c.g. to an arbitrary point on the elevator, and V_4, V_5 are fluid velocities (freestream, behind oblique shock, or behind expansion fan). The position vector is found to be

$$\begin{gathered}\mathbf{r}_e = x\hat{\imath} - [z_{cs} + \tan\delta_e\,(x + x_{cs})]\,\hat{k}\\ -x_{cs} - \tfrac{L_e}{2}\cos\delta_e \le x \le -x_{cs} + \tfrac{L_e}{2}\cos\delta_e\end{gathered} \tag{52}$$

where x_{cs} and z_{cs} are the x and z positions of the midpoint of the elevator referenced to the c.g. As shown in Fig. 1, $x_{cs} = -30ft$ and $z_{cs} = -3.5ft$. For this control surface, outward pointing normal vectors for both the lower and upper surfaces are needed. These normals are computed as

$$\begin{aligned}\mathbf{n}_{e_U} &= -\sin\delta_e\hat{\imath} - \cos\delta_e\hat{k}\\ \mathbf{n}_{e_L} &= \sin\delta_e\hat{\imath} + \cos\delta_e\hat{k}\end{aligned} \tag{53}$$

To evaluate the cross-product in Eq. 51, we use Eq. 52 to obtain

$$\boldsymbol{\omega}\times\mathbf{r}_e = -q\,[z_{cs} + \tan\delta_e\,(x - x_{cs})]\,\hat{\imath} - qx\hat{k} \tag{54}$$

Then, the differential forces on the upper and lower surfaces of the elevator become

$$\begin{aligned}d\mathbf{F}_{e_U} &= [-P_5 - \rho_5 a_5\,\{\mathbf{V}_{e_U}\cdot\mathbf{n}_{e_U}\}]\,\mathbf{n}_{e_U}\,dA_e\\ d\mathbf{F}_{e_L} &= [-P_4 - \rho_4 a_4\,\{\mathbf{V}_{e_L}\cdot\mathbf{n}_{e_L}\}]\,\mathbf{n}_{e_L}\,dA_e\end{aligned} \tag{55}$$

where

$$\mathbf{V}_{e_U}\cdot\mathbf{n}_{e_U} = -\left(u - q\,\{z_{cs} + \tan\delta_e\,(x - x_{cs})\}\right)\sin\delta_e - (w - qx)\cos\delta_e \tag{56}$$

$$\mathbf{V}_{e_L}\cdot\mathbf{n}_{e_L} = \left(u - q\,\{z_{cs} + \tan\delta_e\,(x - x_{cs})\}\right)\sin\delta_e + (w - qx)\cos\delta_e \tag{57}$$

and

$$dA_e = \sec\delta_e dx(1) = \sec\delta_e dx \tag{58}$$

Hence, the upper and lower forces on the elevator can be computed as

$$\mathbf{F}_{e_U} = \int_{-x_{cs}-\frac{L_e}{2}\cos\delta_e}^{-x_{cs}+\frac{L_e}{2}\cos\delta_e} [-P_5 - \rho_5 a_5\,\{\mathbf{V}_{e_U}\cdot\mathbf{n}_{e_U}\}]\left[-\sin\delta_e\hat{\imath} - \cos\delta_e\hat{k}\right]\sec\delta_e dx \tag{59}$$

$$\mathbf{F}_{e_L} = \int_{-x_{cs}-\frac{L_e}{2}\cos\delta_e}^{-x_{cs}+\frac{L_e}{2}\cos\delta_e} [-P_4 - \rho_4 a_4\,\{\mathbf{V}_{e_L}\cdot\mathbf{n}_{e_L}\}]\left[\sin\delta_e\hat{\imath} + \cos\delta_e\hat{k}\right]\sec\delta_e dx \tag{60}$$

Using the steady component of Eq. 56 in Eq. 59 and the steady component of Eq. 57 in Eq. 60, the steady forces on the elevator become

$$\mathbf{F}_{e_U} = \int_{-x_{cs}-\frac{L_e}{2}\cos\delta_e}^{-x_{cs}+\frac{L_e}{2}\cos\delta_e} -P_5\left[-\sin\delta_e\hat{\imath} - \cos\delta_e\hat{k}\right]\sec\delta_e dx \tag{61}$$

$$\mathbf{F}_{e_L} = \int_{-x_{cs}-\frac{L_e}{2}\cos\delta_e}^{-x_{cs}+\frac{L_e}{2}\cos\delta_e} -P_4\left[\sin\delta_e\hat{\imath} + \cos\delta_e\hat{k}\right]\sec\delta_e dx \tag{62}$$

Evaluating these integrals yields

$$\mathbf{F}_{e_U} = -P_5 L_e \left[-\sin\delta_e \hat{\imath} - \cos\delta_e \hat{k}\right] = X_{e_U}\hat{\imath} + Z_{e_U}\hat{k} \tag{63}$$

$$\mathbf{F}_{e_L} = -P_4 L_e \left[\sin\delta_e \hat{\imath} + \cos\delta_e \hat{k}\right] = X_{e_L}\hat{\imath} + Z_{e_L}\hat{k} \tag{64}$$

VII. Flow Analysis

In the preceding analysis, the properties of the flow on the upper, lower, and control surfaces have not been defined. In this section, the properties of the flow will be determined. Specifically, the angles of attack/control deflections at which shock waves or expansion fans are created will be delineated.

By examination of Fig. 1, the following relationships can be determined:

$$\begin{gathered} if \;\; \alpha = \tau_{1,U} \rightarrow freestream : V_1 = V_\infty, \rho_1 = \rho_\infty, a_1 = a_\infty \\ if \;\; \alpha > \tau_{1,U} \rightarrow expansion fan \\ if \;\; \alpha < \tau_{1,U} \rightarrow shock(compression) \end{gathered} \tag{65}$$

The wedge angles associated with the upper surface, for calculation of flow properties behind the shock or expansion fan, are as follows:

$$\begin{gathered} if \;\; \alpha > \tau_{1,U} \rightarrow \theta_{U_{expansion}} = \alpha - \tau_{1,U} \\ if \;\; \alpha < \tau_{1,U} \rightarrow \theta_{U_{shock}} = -\alpha + \tau_{1,U} \end{gathered} \tag{66}$$

The above information is used to determine the flow properties for the upper surface, namely, V_1, ρ_1, and a_1. For lower surface cd, the relationships become

$$\begin{gathered} if \;\; \alpha = -\tau_{1,L} \rightarrow freestream : V_2 = V_\infty, \rho_2 = \rho_\infty, a_2 = a_\infty \\ if \;\; \alpha > -\tau_{1,L} \rightarrow shock(compression) \\ if \;\; \alpha < -\tau_{1,L} \rightarrow expansion fan \end{gathered} \tag{67}$$

Lower surface cd wedge angles are as follows:

$$\begin{gathered} if \;\; \alpha > -\tau_{1,L} \rightarrow \theta_{L_{shock}} = \alpha + \tau_{1,L} \\ if \;\; \alpha < -\tau_{1,L} \rightarrow \theta_{L_{expansion}} = -\alpha - \tau_{1,L} \end{gathered} \tag{68}$$

Physically, for the scramjet engine to work properly, an oblique shock must form on the underside of the vehicle to increase the pressure at the inlet. This effectively places a lower limit on the angle of attack and requires that the angle of attack be such that an oblique shock forms on the underside of the vehicle. Hence, from Eq. 67, the angle of attack must satisfy

$$\alpha > -\tau_{1,L} \tag{69}$$

In other words, $-\tau_{1,L}$ is an absolute lower limit for angle of attack, with the engine on; however, the engine will cease to function before this lower limit is reached. With this limit in place, a bow shock will form on the underside of the vehicle. For lower surface gh, it is necessary to calculate the angle at which the shock exactly impinges on the point g of the engine nacelle. This angle, denoted by $\tau_{bowshock}$, is

$$\tau_{bowshock} = \alpha + \tan^{-1}\left(\frac{L_f \tan\tau_{1,L} + h_i}{L_f}\right) \tag{70}$$

Let the bow shock angle be denoted by β. If $\beta > \tau_{bowshock}$, then the shock misses the point g and the flow properties on lower surface gh are computed using the flow properties behind the oblique shock (V_2, ρ_2, a_2) as the initial conditions. Then, an expansion fan forms at point g. If $\beta \leq \tau_{bowshock}$, then either the shock is on the lip (point g) or the shock is inside the engine inlet. In either case, freestream properties are used to compute V_3, ρ_3, a_3. The following steps are used to determine the flow conditions over surface gh.

1. Calculate shock angle, β, from surface cd.

2. If $\beta > \tau_{bowshock}$, an expansion fan forms at point g and the flow properties behind the oblique shock wave are used as input to the expansion fan flow equations with a wedge angle of $\tau_{1,L}$.

3. If $\beta \leq \tau_{bowshock}$, an expansion fan forms at point g and the freestream flow properties are used as input to the expansion fan flow equations. The wedge angle in this case is only a function of angle of attack. If $\alpha = 0$, then the flow properties in region 3 are freestream, that is, $V_3 = V_\infty, \rho_3 = \rho_\infty$, and $a_3 = a_\infty$. If $\alpha > 0$, a shock forms at point g with the shock angle computed using a wedge angle of α. If $\alpha < 0$, an expansion fan forms on the underside of the engine and flow properties in region 3 are calculated using a wedge angle of $-\alpha$.

For the control surface, the flow behind the leading edge of the elevator is determined by the elevator deflection angle and the angle of attack. More specifically, if $\delta_e = -\alpha$ then both the top and bottom of the elevator experience the freestream. Therefore, $V_4 = V_5 = V_\infty$, $\rho_4 = \rho_5 = \rho_\infty$, and $a_4 = a_5 = a_\infty$. If $\delta_e > -\alpha$, then an expansion fan forms on the top of the elevator, while the bottom of the elevator experiences compression and a shock forms. In either case, the wedge angle is $\alpha + \delta_e$. If $\delta_e < -\alpha$, a shock forms on top of elevator and an expansion fan is on the bottom of the elevator. In this case, the wedge angle for the shock and expansion fan is $-\alpha - \delta_e$.

VIII. Total Forces and Moments

Having determined the forces on each of the surfaces, the moments about the c.g. that each force produces must be determined. To do this, the location of each force on the vehicle must be computed. Figure 3 shows the forces acting on the vehicle. Consider first the upper surface with the forebody and aftbody forces given

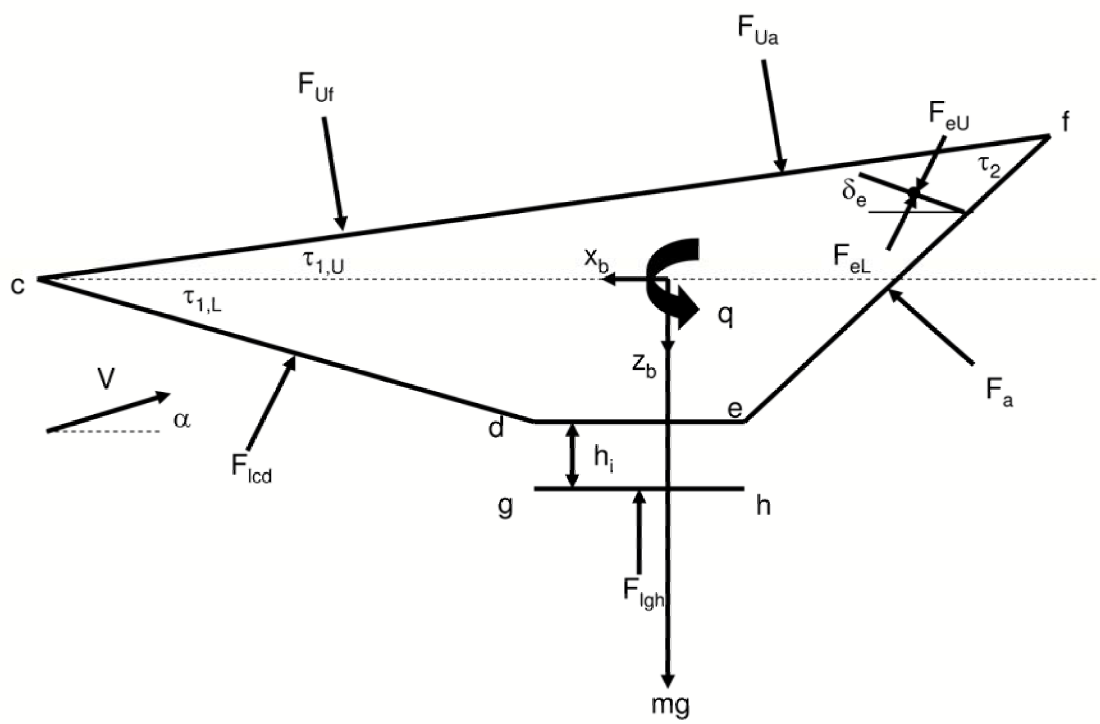

Figure 3. Forces Acting on the Hypersonic Vehicle.

in Eq. 39. The forebody force acts at the point

$$\mathbf{F}_{cf_f} : \left(\frac{\bar{x}_f}{2}, -\frac{\bar{x}_f}{2}\tan\tau_{1,U}\right) \tag{71}$$

while the upper surface afterbody force acts at

$$\mathbf{F}_{cf_a} : \left(-\frac{\bar{x}_a}{2}, -\left(\bar{x}_f + \frac{\bar{x}_a}{2}\right) \tan \tau_{1,U}\right) \tag{72}$$

For the lower surfaces, the forces act at

$$\mathbf{F}_{cd} : \left(\bar{x}_f - \frac{L_f}{2}, \frac{L_f}{2} \tan \tau_{1,L}\right) \tag{73}$$

$$\mathbf{F}_{gh} : \left(\bar{x}_f - L_f - \frac{L_n}{2}, L_f \tan \tau_{1,L} + h_i\right) \tag{74}$$

The force due to the external nozzle acts at a point given by

$$\mathbf{F}_{ef} : \left(\bar{x}_f - L_f - L_n - \bar{x}_{\mathbf{F}_{ef}} \cos\left(\tau_{1,U} + \tau_2\right), L_f \tan \tau_{1,L} - \bar{x}_{\mathbf{F}_{ef}} \sin\left(\tau_{1,U} + \tau_2\right)\right) \tag{75}$$

where $\bar{x}_{\mathbf{F}_{ef}}$ is the x point of the center of mass of the pressure distribution on the rear ramp assuming an axis system centered at point e with x axis pointing along line ef and the z axis pointing up. In other words, the center of mass of the pressure distribution was computed in a local coordinate frame (local to surface ef). Then, this distance was referenced to the c.g. of the vehicle. In Eq. 75, $\bar{x}_{\mathbf{F}_{ef}}$ is given by

$$\bar{x}_{\mathbf{F}_{ef}} = \frac{L_a}{\cos\left(\tau_{1,U} + \tau_2\right) \ln \frac{P_e}{P_\infty}} \left[1 - \frac{P_\infty}{P_e - P_\infty} \ln \frac{P_e}{P_\infty}\right] \tag{76}$$

The elevator force acts at

$$\mathbf{F}_e : \left(-x_{cs}, -z_{cs}\right) \tag{77}$$

Equations 39, 40, 44, 63, and 64 give the axial and normal force components of the forces acting on the vehicle. Now, the moments due to these forces can be calculated. Positive moment is defined as clockwise or the direction that tends to increase angle of attack, while negative moment is defined as counter-clockwise. The moment arms are given by Eqs. 71 - 75, 77.

$$M_{cf_f} = -\frac{P_1 \bar{x}_f^2}{2} \left(1 - \tan^2 \tau_{1,U}\right) \tag{78}$$

$$M_{cf_a} = -P_1 \bar{x}_a \tan^2 \tau_{1,U} \left(\bar{x}_f + \frac{\bar{x}_a}{2}\right) + P_1 \frac{\bar{x}_a^2}{2} \tag{79}$$

$$M_{cd} = -P_2 L_f \tan \tau_{1,L} \left(\frac{L_f \tan \tau_{1,L}}{2}\right) + P_2 L_f \left(\bar{x}_f - \frac{L_f}{2}\right) \tag{80}$$

$$M_{gh} = P_3 L_n \left(\bar{x}_f - L_f - \frac{L_n}{2}\right) \tag{81}$$

$$M_{ef} = \pm X_{ef} \left[L_f \tan \tau_{1,L} - \bar{x}_{\mathbf{F}_{ef}} \sin\left(\tau_{1,U} + \tau_2\right)\right] - Z_{ef} \left[\bar{x}_f - L_f - L_n - \bar{x}_{\mathbf{F}_{ef}} \cos\left(\tau_{1,U} + \tau_2\right)\right] \tag{82}$$

where $\bar{x}_{\mathbf{F}_{ef}}$ is given in Eq. 76 and X_{ef}, Z_{ef} are expressed in Eq. 44. The sign on the first component of M_{ef} will depend on the vehicle's geometry. The rule used to determine the sign is as follows:

$$\begin{array}{ll} + \;\; if & L_f \tan \tau_{1,L} - \bar{x}_{\mathbf{F}_{ef}} \sin\left(\tau_{1,U} + \tau_2\right) > 0 \\ - \;\; if & L_f \tan \tau_{1,L} - \bar{x}_{\mathbf{F}_{ef}} \sin\left(\tau_{1,U} + \tau_2\right) \leq 0 \end{array} \tag{83}$$

The moments produced by the elevator are

$$M_{e_U} = -P_5 L_e \sin \delta_e z_{cs} + P_5 L_e \cos \delta_e x_{cs} \tag{84}$$

$$M_{e_L} = P_4 L_e \sin \delta_e z_{cs} - P_4 L_e \cos \delta_e x_{cs} \tag{85}$$

The total aerodynamic forces and moments on the vehicle are

$$X_{total} = X_{cf_f} + X_{cf_a} + X_{cd} + X_{gh} + X_{ef} + X_{e_L} + X_{e_U} \tag{86}$$

$$Z_{total} = Z_{cf_f} + Z_{cf_a} + Z_{cd} + Z_{gh} + Z_{ef} + Z_{e_L} + Z_{e_U} \tag{87}$$

$$M_{total} = M_{cf_f} + M_{cf_a} + M_{cd} + M_{gh} + M_{ef} + M_{e_L} + M_{e_U} \tag{88}$$

IX. Stability Derivatives

The stability derivatives of interest for this vehicle configuration are $C_{Z\alpha}$, $C_{x\alpha}$, $C_{M\alpha}$, C_{Zq}, and C_{Mq}. These computations are complex and, in an attempt to simplify them, the following notation will be used when necessary:

$$\begin{aligned} L_{ef_2} &= \frac{(L_a-\bar{x}_a)^2}{2} - \frac{\bar{x}_a^2}{2} \\ L_{ef_3} &= \frac{(L_a-\bar{x}_a)^3}{3} + \frac{\bar{x}_a^3}{3} \\ L_{ef_4} &= \frac{(L_a-\bar{x}_a)^4}{4} - \frac{\bar{x}_a^4}{4} \\ L_{ef_5} &= \frac{(L_a-\bar{x}_a)^5}{5} + \frac{\bar{x}_a^5}{5} \end{aligned} \tag{89}$$

$$\begin{aligned} A_1 &= \rho_e a_e + \frac{(\rho_e-\rho_\infty)(a_e-a_\infty)(L_a-\bar{x}_a)^2}{L_a^2} - \frac{a_e}{L_a}(\rho_e-\rho_\infty)(L_a-\bar{x}_a) - \frac{\rho_e}{L_a}(a_e-a_\infty)(L_a-\bar{x}_a) \\ A_2 &= \frac{a_e}{L_a}(\rho_e-\rho_\infty) + \frac{\rho_e}{L_a}(a_e-a_\infty) - \frac{2}{L_a^2}(\rho_e-\rho_\infty)(a_e-a_\infty)(L_a-\bar{x}_a) \\ A_3 &= \frac{(\rho_e-\rho_\infty)(a_e-a_\infty)}{L_a^2} \end{aligned} \tag{90}$$

$$\begin{aligned} f_1 &= \tan(\tau_{1,U}+\tau_2) \\ f_2 &= L\tan\tau_{1,U} \\ f_3 &= \sin(\tau_{1,U}+\tau_2) \\ f_4 &= \cos(\tau_{1,U}+\tau_2) \end{aligned} \tag{91}$$

Additionally, the pressure on the rear ramp of the vehicle (surface ef) is given by Eq. 41, which is a function of the distance moved along the rear ramp. Thus, this pressure distribution is not constant along the surface. At point e, the pressure is given by the engine exit properties (P_e), while at point f the pressure is the freestream pressure (P_∞). An obvious choice for the temperature distribution on the rear ramp, T_{ef}, is a form similar to the pressure, so that

$$T_{ef}(s) \approx \frac{T_e}{1 + \frac{s}{\frac{L_a}{\cos(\tau_{1,U}+\tau_2)}}\left(\frac{T_e}{T_\infty}-1\right)} \tag{92}$$

Then, the speed of sound and density can be calculated from the definition of speed of sound and the perfect gas law:

$$\begin{aligned} a_{ef} &= \sqrt{\gamma R T_{ef}} \\ \rho_{ef} &= \frac{P_{ef}}{RT_{ef}} \end{aligned} \tag{93}$$

Both a_{ef} and ρ_{ef} factor into the stability derivative calculations. Unfortunately, the expressions in Eq. 93, when used in the stability derivative calculations do not allow determination of a closed-form solution. In order to facilitate a closed-form solution, the following approximations for the speed of sound and density on the rear ramp will be used in the stability derivative calculations:

$$\begin{aligned} \rho_{ef} &= \frac{(\rho_\infty-\rho_e)}{-L_a}(x-\{L_a-\bar{x}_a\}) + \rho_e \\ a_{ef} &= \frac{(a_\infty-a_e)}{-L_a}(x-\{L_a-\bar{x}_a\}) + a_e \end{aligned} \tag{94}$$

where

$$-\bar{x}_a \le x \le L_a - \bar{x}_a \tag{95}$$

These are first-order approximations, which capture the boundary conditions.

A. α Derivative of Z-Force Coefficient

To compute the change in Z-force component due to a change in angle of attack, the infinitesimal force expression that contains the vertical velocity perturbation, w, must be integrated. Thus,

$$\begin{aligned} (C_Z)_w = \frac{1}{q_\infty S}\int (d\mathbf{F})_{z-w} &= \frac{1}{q_\infty S}\left[\int_{-\bar{x}_a}^{\bar{x}_f} (d\mathbf{F}_{cf})_{z-w} + \int_{\bar{x}_f-L_f}^{\bar{x}_f} (d\mathbf{F}_{cd})_{z-w}\right] \\ &+ \frac{1}{q_\infty S}\left[\int_{\bar{x}_f-L_f-L_n}^{\bar{x}_f-L_f} (d\mathbf{F}_{gh})_{z-w} + \int_{-\bar{x}_a}^{L_a-\bar{x}_a} (d\mathbf{F}_{ef})_{z-w}\right] \end{aligned} \tag{96}$$

where $\int (d\mathbf{F}_{cf})_{z-w}$ is the differential force, on the upper surface, in the z direction, due to w motion and likewise for the other surfaces. Using the appropriate differential force elements from Eq. 25 in Eq. 96 produces

$$\begin{aligned}(C_Z)_w = \frac{1}{q_\infty S}\left[\int_{-\bar{x}_a}^{\bar{x}_f} -\rho_1 a_1 w \cos\tau_{1,U} dx - \int_{\bar{x}_f - L_f}^{\bar{x}_f} \rho_2 a_2 w \cos\tau_{1,L} dx\right] \\ + \frac{1}{q_\infty S}\left[\int_{\bar{x}_f - L_f - L_n}^{\bar{x}_f - L_f} -\rho_3 a_3 w dx + \int_{\bar{x}_a}^{L_a - \bar{x}_a} -\rho_{ef} a_{ef} w \cos(\tau_{1,U} + \tau_2) dx\right]\end{aligned} \tag{97}$$

Performing the integrations yields

$$\begin{aligned}(C_Z)_w = w\frac{1}{q_\infty S}\left[-\rho_1 a_1 \cos\tau_{1,U}(\bar{x}_f + \bar{x}_a) - \rho_2 a_2 \cos\tau_{1,L} L_f - \rho_3 a_3 L_n\right] \\ -w\frac{\cos(\tau_{1,U} + \tau_2)}{q_\infty S}\left[A_1 L_a + A_2 L_{ef_2} + A_3 L_{ef_3}\right]\end{aligned} \tag{98}$$

If $\frac{w}{V_\infty} \ll 1$ then $\frac{w}{V_\infty} \approx \alpha$. Thus, $w = V_\infty \alpha$. Therefore,

$$\begin{aligned}\frac{\partial C_Z}{\partial \alpha} = \frac{V_\infty}{q_\infty S}\left[-\rho_1 a_1 \cos\tau_{1,U} L - \rho_2 a_2 \cos\tau_{1,L} L_f - \rho_3 a_3 L_n\right] \\ -\frac{V_\infty \cos(\tau_{1,U} + \tau_2)}{q_\infty S}\left[A_1 L_a + A_2 L_{ef_2} + A_3 L_{ef_3}\right]\end{aligned} \tag{99}$$

B. α Derivative of X-Force Coefficient

The change in X-force due to a change in angle of attack can be calculated using

$$\begin{aligned}(C_X)_w = \frac{1}{q_\infty S}\int (d\mathbf{F})_{x-w} = \frac{1}{q_\infty S}\left[\int_{-\bar{x}_a}^{\bar{x}_f} (d\mathbf{F}_{cf})_{x-w} + \int_{\bar{x}_f - L_f}^{\bar{x}_f} (d\mathbf{F}_{cd})_{x-w}\right] \\ + \frac{1}{q_\infty S}\left[\int_{\bar{x}_f - L_f - L_n}^{\bar{x}_f - L_f} (d\mathbf{F}_{gh})_{x-w} + \int_{-\bar{x}_a}^{L_a - \bar{x}_a} (d\mathbf{F}_{ef})_{x-w}\right]\end{aligned} \tag{100}$$

Substituting in the differential forces yields

$$\begin{aligned}(C_X)_w = \frac{1}{q_\infty S}\left[\int_{-\bar{x}_a}^{\bar{x}_f} \rho_1 a_1 w \sin\tau_{1,U} dx - \int_{\bar{x}_f - L_f}^{\bar{x}_f} \rho_2 a_2 w \sin\tau_{1,L} dx\right] \\ + \frac{1}{q_\infty S}\left[\int_{\bar{x}_f - L_f - L_n}^{\bar{x}_f - L_f} -\rho_3 a_3 w 0 dx + \int_{-\bar{x}_a}^{L_a - \bar{x}_a} \rho_{ef} a_{ef} w \sin(\tau_{1,U} + \tau_2) dx\right]\end{aligned} \tag{101}$$

Performing the integrations, simplifying, and assuming that $\frac{w}{V_\infty} \ll 1$ so that $\frac{w}{V_\infty} \approx \alpha$ gives

$$\frac{\partial C_X}{\partial \alpha} = \frac{1}{q_\infty S} V_\infty \left[\rho_1 a_1 \sin\tau_{1,U} L - \rho_2 a_2 \sin\tau_{1,L} L_f\right] + \frac{V_\infty \sin(\tau_{1,U} + \tau_2)}{q_\infty S}\left[A_1 L_a + A_2 L_{ef_2} + A_3 L_{ef_3}\right] \tag{102}$$

C. α Derivative of Pitching Moment Coefficient

For $C_{M\alpha}$, first find the contribution to the pitching moment due to a velocity w:

$$\begin{aligned}(C_M)_w &= \frac{1}{q_\infty S\bar{c}}\left[\int z(d\mathbf{F})_{x-w} - \int x(d\mathbf{F})_{z-w}\right] \\ &= \frac{1}{q_\infty S\bar{c}} z\left[\int_{-\bar{x}_a}^{\bar{x}_f} (d\mathbf{F}_{cf})_{x-w} + \int_{\bar{x}_f - L_f}^{\bar{x}_f} (d\mathbf{F}_{cd})_{x-w} + \int_{\bar{x}_f - L_f - L_n}^{\bar{x}_f - L_f} (d\mathbf{F}_{gh})_{x-w} + \int_{-\bar{x}_a}^{L_a - \bar{x}_a} (d\mathbf{F}_{ef})_{x-w}\right] \\ &- \frac{1}{q_\infty S\bar{c}} x\left[\int_{-\bar{x}_a}^{\bar{x}_f} (d\mathbf{F}_{cf})_{z-w} + \int_{\bar{x}_f - L_f}^{\bar{x}_f} (d\mathbf{F}_{cd})_{z-w} + \int_{\bar{x}_f - L_f - L_n}^{\bar{x}_f - L_f} (d\mathbf{F}_{gh})_{z-w} + \int_{-\bar{x}_a}^{L_a - \bar{x}_a} (d\mathbf{F}_{ef})_{z-w}\right]\end{aligned} \tag{103}$$

Substituting the appropriate expressions, performing the integrations, simplifying, and assuming that $\frac{w}{V_\infty} \ll 1$ so that $\frac{w}{V_\infty} \approx \alpha$ yields

$$\begin{aligned} \frac{\partial C_M}{\partial \alpha} &= \frac{1}{q_\infty S \bar{c}} \frac{1}{2} V_\infty \left[-\rho_1 a_1 \tan \tau_{1,U} \sin \tau_{1,U} L^2 + \rho_2 a_2 \tan \tau_{1,L} \sin \tau_{1,L} L_f^2 \right] \\ &+ \frac{V_\infty \sin(\tau_{1,U} + \tau_2)}{q_\infty S \bar{c}} \left[f_1 \left(A_1 L_{ef_2} + A_2 L_{ef_3} + A_3 L_{ef_4} \right) + \left(f_1 \bar{x}_a - f_2 \right) \left(A_1 L_a + A_2 L_{ef_2} + A_3 L_{ef_3} \right) \right] \\ &+ \frac{1}{q_\infty S \bar{c}} \frac{1}{2} V_\infty \left[\rho_1 a_1 \cos \tau_{1,U} \left(\bar{x}_f^2 - \bar{x}_a^2 \right) + \rho_2 a_2 \cos \tau_{1,L} L_f \left(2\bar{x}_f - L_f \right) - \rho_3 a_3 L_n \left(L_n - 2\bar{x}_f + 2L_f \right) \right] \\ &+ \frac{V_\infty \cos(\tau_{1,U} + \tau_2)}{q_\infty S \bar{c}} \left[A_1 L_{ef_2} + A_2 L_{ef_3} + A_3 L_{ef_4} \right] \end{aligned} \tag{104}$$

D. q Derivative of Z-Force Coefficient

The Z-force coefficient due to a pitching motion is

$$\begin{aligned} (C_Z)_q = \frac{1}{q_\infty S} \int (d\mathbf{F})_{z-q} &= \frac{1}{q_\infty S} \left[\int_{-\bar{x}_a}^{\bar{x}_f} (d\mathbf{F}_{cf})_{z-q} + \int_{\bar{x}_f - L_f}^{\bar{x}_f} (d\mathbf{F}_{cd})_{z-q} \right] \\ &+ \frac{1}{q_\infty S} \left[\int_{\bar{x}_f - L_f - L_n}^{\bar{x}_f - L_f} (d\mathbf{F}_{gh})_{z-q} + \int_{-\bar{x}_a}^{L_a - \bar{x}_a} (d\mathbf{F}_{ef})_{z-q} \right] \end{aligned} \tag{105}$$

Performing the integrations and simplifying yields

$$\begin{aligned} \frac{\partial C_Z}{\partial q} &= \frac{1}{q_\infty S} \rho_1 a_1 \frac{L}{2} \left[\cos \tau_{1,U} - \tan \tau_{1,U} \sin \tau_{1,U} L \right] \\ &+ \frac{1}{q_\infty S} \rho_2 a_2 \frac{L_f}{2} \left[\cos \tau_{1,L} \left(2\bar{x}_f - L_f \right) - \tan \tau_{1,L} \sin \tau_{1,L} L_f \right] \\ &+ \frac{1}{q_\infty S} \rho_3 a_3 \frac{L_n}{2} \left[-L_n + 2\bar{x}_f - 2L_f \right] \\ &+ \frac{1}{q_\infty S} \left[\left(f_1 f_3 - f_4 \right) \left(A_1 L_{ef_2} + A_2 L_{ef_3} + A_3 L_{ef_4} \right) + \left(f_1 f_3 \bar{x}_a - f_2 f_3 \right) \left(A_1 L_a + A_2 L_{ef_2} + A_3 L_{ef_3} \right) \right] \end{aligned} \tag{106}$$

E. q Derivative of Pitching Moment Coefficient

The pitching moment due to a pitch rate can be calculated using

$$\begin{aligned} (C_M)_q &= \frac{1}{q_\infty S \bar{c}} \int z\, (d\mathbf{F})_{x-q} - \int x\, (d\mathbf{F})_{z-q} \\ &= \frac{1}{q_\infty S \bar{c}} z \left[\int_{-\bar{x}_a}^{\bar{x}_f} (d\mathbf{F}_{cf})_{x-q} + \int_{\bar{x}_f - L_f}^{\bar{x}_f} (d\mathbf{F}_{cd})_{x-q} + \int_{\bar{x}_f - L_f - L_n}^{\bar{x}_f - L_f} (d\mathbf{F}_{gh})_{x-q} + \int_{-\bar{x}_a}^{L_a - \bar{x}_a} (d\mathbf{F}_{ef})_{x-q} \right] \\ &- \frac{1}{q_\infty S \bar{c}} x \left[\int_{-\bar{x}_a}^{\bar{x}_f} (d\mathbf{F}_{cf})_{z-q} + \int_{\bar{x}_f - L_f}^{\bar{x}_f} (d\mathbf{F}_{cd})_{z-q} + \int_{\bar{x}_f - L_f - L_n}^{\bar{x}_f - L_f} (d\mathbf{F}_{gh})_{z-q} + \int_{-\bar{x}_a}^{L_a - \bar{x}_a} (d\mathbf{F}_{ef})_{z-q} \right] \end{aligned} \tag{107}$$

Substituting the appropriate expressions, performing the integrations, and simplifying produces

$$
\begin{aligned}
\frac{\partial C_M}{\partial q} = & -\frac{1}{q_\infty S\bar{c}}\rho_1 a_1 \tan^2\tau_{1,U}\left[\tan\tau_{1,U}\sin\tau_{1,U}\frac{L^3}{3}+\frac{\cos\tau_{1,U}}{6}\left(-\bar{x}_f^3+2\bar{x}_a^3+3\bar{x}_a^2\bar{x}_f\right)\right] \\
& -\frac{1}{q_\infty S\bar{c}}\rho_2 a_2 \tan^2\tau_{1,L}\left[\tan\tau_{1,L}\sin\tau_{1,L}\frac{L_f^3}{3}-\cos\tau_{1,L}\frac{L_f}{2}\left(2\bar{x}_f-L_f\right)\right] \\
& -\frac{f_1}{q_\infty S\bar{c}}\left[\left(A_1L_{ef_3}+A_2L_{ef_4}+A_3L_{ef_5}\right)\left(f_1^2f_3+f_1f_4\right)\right] \\
& -\frac{f_1}{q_\infty S\bar{c}}\left[\left(A_1L_{ef_2}+A_2L_{ef_3}+A_3L_{ef_4}\right)\left(2f_1^2f_3\bar{x}_a-2f_1f_2f_3-f_2f_4+f_1f_4\bar{x}_a\right)\right] \\
& -\frac{f_1}{q_\infty S\bar{c}}\left[\left(A_1L_a+A_2L_{ef_2}+A_3L_{ef_3}\right)\left(f_1^2f_3\bar{x}_a^2-2f_1f_2f_3\bar{x}_a+f_2^2f_3\right)\right] \\
& -\frac{1}{q_\infty S\bar{c}}\rho_1 a_1\left[\frac{\tan\tau_{1,U}\sin\tau_{1,U}}{6}\left(-\bar{x}_f^3+3\bar{x}_a^2\bar{x}_f+2\bar{x}_a^3\right)+\frac{\cos\tau_{1,U}}{3}\left(\bar{x}_f^3+\bar{x}_a^3\right)\right] \\
& -\frac{1}{q_\infty S\bar{c}}\rho_2 a_2\left[\frac{\tan\tau_{1,L}\sin\tau_{1,L}}{6}L_f^2\left(2L_f-3\bar{x}_f\right)+\cos\tau_{1,L}\frac{L_f}{3}\left(3\bar{x}_f^2-3\bar{x}_fL_f+L_f^2\right)\right] \\
& -\frac{1}{q_\infty S\bar{c}}\rho_3 a_3\left[\frac{\left(\bar{x}_f-L_f\right)^3}{3}-\frac{\left(\bar{x}_f-L_f-L_n\right)^3}{3}\right] \\
& -\frac{1}{q_\infty S\bar{c}}\left[\left(A_1L_{ef_3}+A_2L_{ef_4}+A_3L_{ef_5}\right)\left(f_1f_3+f_4\right)+\left(A_1L_{ef_2}+A_2L_{ef_3}+A_3L_{ef_4}\right)\left(f_1f_3\bar{x}_a-f_2f_3\right)\right]
\end{aligned}
\tag{108}
$$

X. Engine

For hypersonic flight, the propulsion system necessary to produce the required thrust is either a rocket or a supersonic combustion ramjet (scramjet). The advantage of a scramjet over a rocket is that the scramjet is an airbreathing propulsion system, thus eliminating the need to carry the oxidizer onboard the vehicle. This, in turn, allows for increased payload. The scramjet model used in this work is identical to that used by Chavez and Schmidt.[5] Engine inlet conditions are primarily determined by the flow behind the oblique shock. The scramjet consists of 3 sections; a diffuser, a combustor, and an internal nozzle. The flow through the diffuser and nozzle is assumed to be isentropic, quasi-one-dimensional, while the flow through the combustor is assumed to be quasi-one-dimensional in a constant area duct with heat addition. The working fluid in the engine is assumed to be a perfect gas with constant specific heats. There are two control variables which affect the engine: diffuser area ratio, $\bar{A}_D$, and temperature addition in the combustor, ΔT_0. Figure 4 shows the engine model.

Inlet conditions to the diffuser, which are inlet conditions to the engine module, are determined from the flow analysis of the vehicle's lower forebody and vehicle geometry. The lower forebody flow is turned parallel to surface cd. To determine the engine inlet conditions, the flow must be turned parallel to the engine, with the turning angle given by $\tau_{1,L}$. Oblique shock relations are used to determine the engine inlet flow properties, with the flow properties in region 2 being the input, and $M_{e_{in}}, P_{e_{in}}$, and $T_{e_{in}}$ (the engine inlet flow properties) being the outputs of the flow calculations. By turning the flow to be parallel to the engine, a force and moment is imparted on the vehicle. The force is given by

$$
\begin{aligned}
F_{x_{inlet}} &= \gamma M_2^2 P_2\left(1-\cos\left(\tau_{1,L}+\alpha\right)\right)\frac{\frac{A_e}{b}}{A_dA_n} \\
F_{z_{inlet}} &= \gamma M_2^2 P_2\sin\left(\tau_{1,L}+\alpha\right)\frac{\frac{A_e}{b}}{A_dA_n}
\end{aligned}
\tag{109}
$$

This force acts at point d in Figure 1. Hence, the force acts at

$$
\mathbf{r}_{inlet} = \begin{bmatrix} \bar{x}_f-L_f & L_f\tan\tau_{1,L} \end{bmatrix} \tag{110}
$$

Thus, the moment produced is given by

$$
M_{inlet} = L_f\tan\tau_{1,L}F_{x_{inlet}}-\left(\bar{x}_f-L_f\right)F_{z_{inlet}} \tag{111}
$$

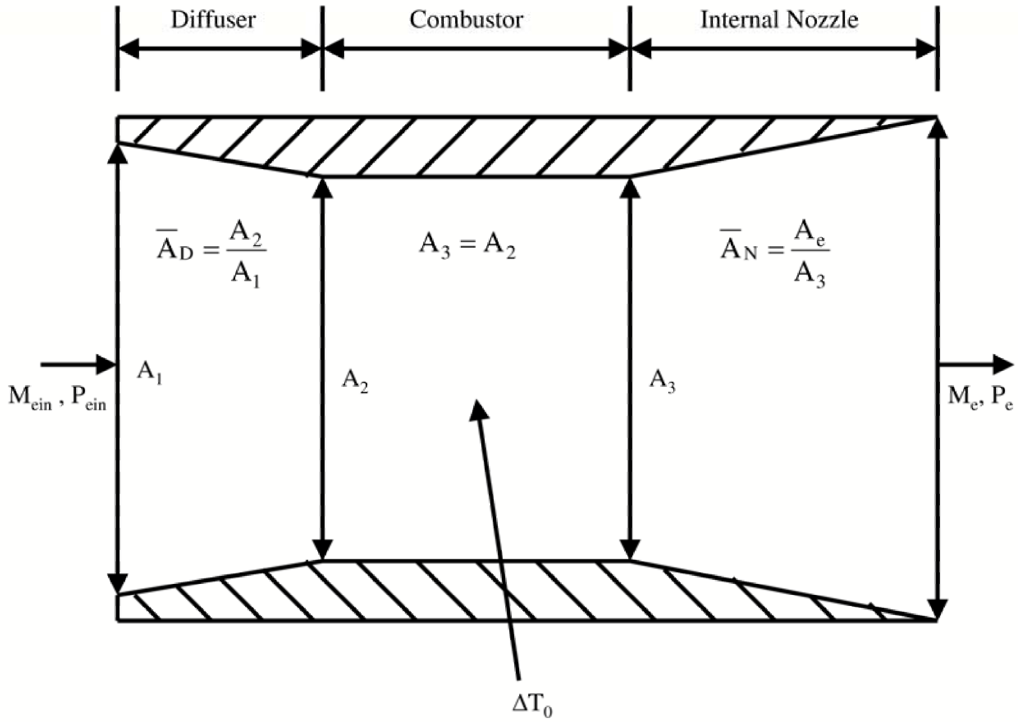

Figure 4. Scramjet Engine.

Now, the flow must be propagated through each section of the scramjet to determine the engine output properties. For the diffuser, continuity is applied to determine the Mach number at the diffuser exit/combustor inlet:

$$\frac{\left[1+\frac{1}{2}\left(\gamma-1\right)M_c^2\right]^{\left(\frac{\gamma+1}{\gamma-1}\right)}}{M_c^2} = \bar{A}_D^2 \frac{\left[1+\frac{1}{2}\left(\gamma-1\right)M_{e_{in}}^2\right]^{\left(\frac{\gamma+1}{\gamma-1}\right)}}{M_{e_{in}}^2} \tag{112}$$

where M_c is the Mach number at the combustor inlet and $M_{e_{in}}$ is the Mach number at the engine inlet. The pressures and temperatures at the combustor inlet are given by

$$P_c = P_{e_{in}} \left[\frac{1+\frac{1}{2}\left(\gamma-1\right)M_{e_{in}}^2}{1+\frac{1}{2}\left(\gamma-1\right)M_c^2}\right]^{\frac{\gamma}{\gamma-1}} \tag{113}$$

$$T_c = T_{e_{in}} \left[\frac{1+\frac{1}{2}\left(\gamma-1\right)M_{e_{in}}^2}{1+\frac{1}{2}\left(\gamma-1\right)M_c^2}\right] \tag{114}$$

where P_c and T_c are the pressure and temperature at the combustor inlet, and $P_{e_{in}}$ and $T_{e_{in}}$ are the pressure and temperature at the engine inlet. For the combustor, the exit Mach number, temperature, and pressure are calculated using

$$\frac{M_n^2\left[1+\frac{1}{2}\left(\gamma-1\right)M_n^2\right]}{\left(\gamma M_n^2+1\right)^2} = \frac{M_c^2\left[1+\frac{1}{2}\left(\gamma-1\right)M_c^2\right]}{\left(\gamma M_c^2+1\right)^2} + \frac{M_c^2}{\left(\gamma M_c^2+1\right)^2}\frac{\Delta T_0}{T_c} \tag{115}$$

$$P_n = P_c \frac{\gamma M_c^2+1}{\gamma M_n^2+1} \tag{116}$$

$$T_n = T_c \left(\frac{\gamma M_c^2+1}{\gamma M_n^2+1}\frac{M_n}{M_c}\right)^2 \tag{117}$$

where M_n, P_n, and T_n are the Mach number, pressure, and temperature at the inlet to the nozzle, respectively and ΔT_0 is the increase in total temperature across the combustor due to the combustion of fuel. For the nozzle, the exit properties are

$$\frac{\left[1+\frac{1}{2}\left(\gamma-1\right)M_e^2\right]^{\left(\frac{\gamma+1}{\gamma-1}\right)}}{M_e^2} = \bar{A}_N^2 \frac{\left[1+\frac{1}{2}\left(\gamma-1\right)M_n^2\right]^{\left(\frac{\gamma+1}{\gamma-1}\right)}}{M_n^2} \tag{118}$$

where M_e is the Mach number at the engine exit. The pressures and temperatures at the engine exit are given by

$$P_e = P_n \left[\frac{1+\frac{1}{2}\left(\gamma-1\right)M_n^2}{1+\frac{1}{2}\left(\gamma-1\right)M_e^2}\right]^{\frac{\gamma}{\gamma-1}} \tag{119}$$

$$T_e = T_n \left[\frac{1+\frac{1}{2}\left(\gamma-1\right)M_n^2}{1+\frac{1}{2}\left(\gamma-1\right)M_e^2}\right] \tag{120}$$

where M_e, P_e, and T_e are the Mach number, pressure, and temperature at the engine exit, respectively, and $\bar{A}_N$ is the internal nozzle ratio defined as the ratio of the nozzle exit area to the nozzle inlet area.

The thrust per unit width of the engine module is given by[4]

$$T = \dot{m}_o\left(V_e - V_\infty\right) + \left(p_e - p_\infty\right)\frac{A_e}{b} - \left(p_2 - p_\infty\right)\frac{\frac{A_e}{b}}{A_D A_N} \tag{121}$$

where T is the engine thrust, $\dot{m}_o$ is the mass flow through the engine, V_e, p_e are the flow velocity and pressure at the engine exit, V_2, p_2 are the flow velocity and pressure at the engine inlet, p_∞ is the freestream pressure, A_e/b is the exit area per unit width, A_D is the diffuser area ratio, and A_N is the nozzle area ratio. The moment produced by the thrust force is

$$M_{engine} = \left(L_f \tan\tau_{1,L} + \frac{h_i}{2}\right)T \tag{122}$$

The mass flow through the engine is a function of the shock angle. Essentially, in order to calculate $\dot{m}_o$, it is necessary to calculate how much mass flow the engine captures. Figure 5 sets up the geometry for this calculation. It is assumed that the vehicle is operating such that a shock forms off the forebody surface. Let the engine inlet capture area be denoted by A_0 and the spill area be denoted by A_s. When a shock forms, use Figure 5 to compute the capture area, which becomes

$$\begin{aligned} A_0 &= \tfrac{\left(L_f - 2\{L_f - (L_f\tan\tau_{1,L} + h_i)\cot(\beta_{lcd}-\alpha)\}\right)}{\cos(\beta_{lcd}-\alpha)}\sin\beta_{lcd} \\ A_1 &= \tfrac{\left(L_f\tan\tau_{1,L}+h_i\right)}{\sin\theta_1}\sin\left(\alpha+\theta_1\right) \end{aligned} \tag{123}$$

where β_{lcd} is the shock angle for lower surface cd and θ_1 is defined as

$$\theta_1 = \tan^{-1}\left(\frac{L_f\tan\tau_{1,L} + h_i}{L_f}\right) \tag{124}$$

The mass flow through the engine becomes

$$\dot{m}_o = P_\infty M_\infty \sqrt{\frac{\gamma}{RT_\infty}}A_0 \tag{125}$$

With the inclusion of the stability derivatives, the engine inlet turning force and moment, and the thrust and resulting moment, the total aerodynamic forces and moments on the vehicle are

$$X_{total} = X_{cf_f} + X_{cf_a} + X_{cd} + X_{gh} + X_{ef} + X_{e_L} + X_{e_U} + q_\infty S\frac{\partial C_X}{\partial\alpha}\alpha + F_{x_{inlet}} \tag{126}$$

$$Z_{total} = Z_{cf_f} + Z_{cf_a} + Z_{cd} + Z_{gh} + Z_{ef} + Z_{e_L} + Z_{e_U} + q_\infty S\frac{\partial C_Z}{\partial\alpha}\alpha + q_\infty S\frac{\partial C_Z}{\partial q}\frac{q\bar{c}}{2V_\infty} + F_{z_{inlet}} \tag{127}$$

$$M_{total} = M_{cf_f} + M_{cf_a} + M_{cd} + M_{gh} + M_{ef} + M_{e_L} + M_{e_U} + q_\infty S\bar{c}\frac{\partial C_M}{\partial\alpha}\alpha + q_\infty S\bar{c}\frac{\partial C_M}{\partial q}\frac{q\bar{c}}{2V_\infty} + M_{inlet} + M_{engine} \tag{128}$$

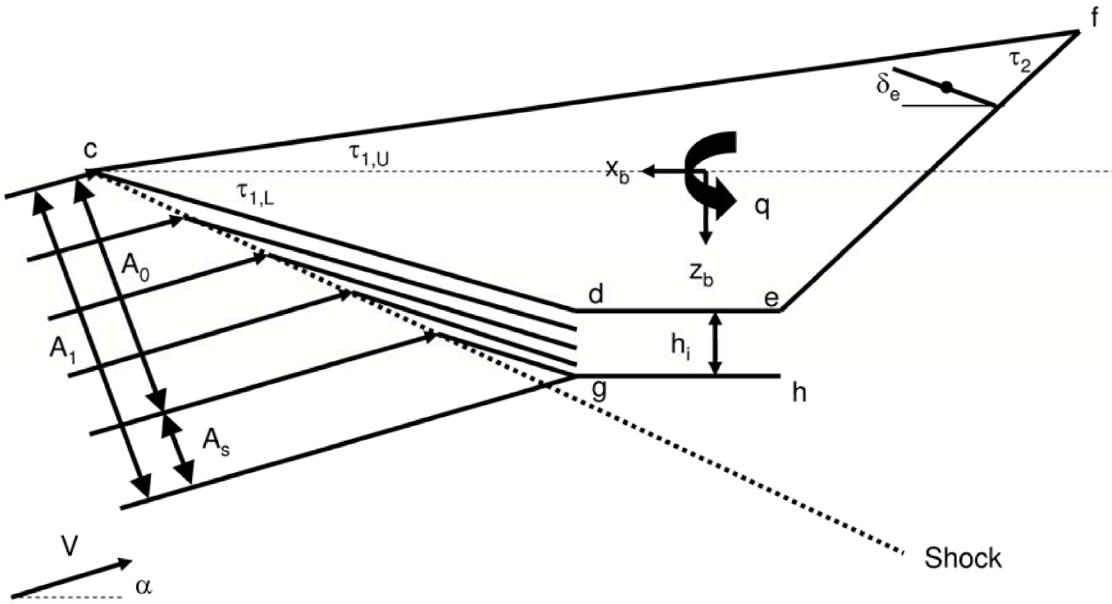

Figure 5. Geometry for Mass Flow Capture Area.

XI. Aeroelastic Effects

Thus far, only a rigid body has been considered. In fact, the vehicle is long and slender and thus highly flexible. This structural bending affects downstream flow and should be incorporated into the model. In this section, the aeroelastic effects are included in the analysis. Piston theory is still used to determine the pressure distribution on the surfaces of the vehicle and much of the analysis already presented can be easily adapted to included these additional effects.

A. Aeroelastic Model

In order to develop the aeroelastic model, a few assumptions are made. First, the flexible vehicle is modelled as a cantilever beam fixed at the c.g. (only the forward section of the vehicle is currently considered flexible). Second, the beams are assumed to have constant mass density, area, and flexural rigidity (EI), where EI is chosen to give the desired natural frequency of vibration. Also, it is assumed that the flexible effects only perturb the surface velocities in the z (normal) direction. This assumption is justified using the small angle approximation, i.e., the deflection of the tip of the beam is small compared to the length of the beam. Lastly, it is assumed that the pressure distribution over the forebody of the vehicle is constant even when the beam is deflected.

The transverse vibrations in the beam satisfy the following partial differential equation:[6]

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \hat{m}\frac{\partial^2 w(x,t)}{\partial t^2} = 0 \tag{129}$$

where $w(x,t)$ describes the position of the beam, E is Young's Modulus, and I is the moment of inertia of the beam cross-section about the y-axis. To solve this, use the method of separation of variables. Assume

$$w(x,t) = \Phi(x)T(t) \tag{130}$$

Substituting the expression for $w(x,t)$ in Eq. 130 into Eq. 129 and simplifying yields

$$\frac{EI}{\hat{m}\Phi(x)}\frac{\partial^4\Phi(x)}{\partial x^4} = -\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} \tag{131}$$

Since the left side of Eq. 131 does not change as time varies, the right side of Eq. 131 must be a constant. Similarly, since the right side of Eq. 131 does not change as x varies, the left side of Eq. 131 must be a constant. Let this constant be ω^2, such that

$$\frac{EI}{\hat{m}\Phi(x)}\frac{\partial^4\Phi(x)}{\partial x^4} = -\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = \omega^2 \tag{132}$$

Utilizing the method of separation of variables, Eq. 132 can be written as two differential equations, one with respect to position and one with respect to time:

$$\frac{\partial^4\Phi(x)}{\partial x^4} - \beta^4\Phi(x) = 0 \tag{133}$$

$$\frac{\partial^2 T(t)}{\partial t^2} + \omega^2 T(t) = 0 \tag{134}$$

where $\beta^4 = \frac{\omega^2\hat{m}}{EI}$. The general solution to Eq. 133 is[6]

$$\Phi(x) = C_1\sin\beta x + C_2\cos\kappa x + C_3\sinh\kappa x + C_4\cosh\kappa x \tag{135}$$

Using the following boundary conditions for the forward beam

$$\begin{array}{cc} \frac{d^2\Phi(x)}{dx^2}|_{x=\bar{x}_f} = 0 & \frac{d^3\Phi(x)}{dx^3}|_{x=\bar{x}_f} = 0 \\ \Phi(x)|_{x=0} = 0 & \frac{d\Phi(x)}{dx}|_{x=0} = 0 \end{array} \tag{136}$$

which state that the bending moment and shear force are zero at the free location ($x = \bar{x}_f$) and the displacement and slope are zero at the fixed location ($x = 0$), along with the modal shape expression, (Eq. 135), and simplifying results in the frequency equation

$$\cos\beta L\cosh\beta L = -1 \tag{137}$$

Eq. 137 has an infinite number of solutions with the first few given by

$$\beta L = 1.8751, 4.6941, 7.8548, 10.9955, 14.1372, \ldots \tag{138}$$

The values of β_r, $r = 1, 2, 3, \ldots$ in Eq. 138 are called the eigenvalues. Corresponding to these eigenvalues, the natural modes of the forward beam are[6]

$$\begin{aligned}\Phi_{f,r}(x) = A_{f,r}&\left[(\sin\beta_{f,r}\bar{x}_f - \sinh\beta_{f,r}\bar{x}_f)(\sin\beta_{f,r}x - \sinh\beta_{f,r}x)\right] \\ +A_{f,r}&\left[(\cos\beta_{f,r}\bar{x}_f + \cosh\beta_{f,r}\bar{x}_f)(\cos\beta_{f,r}x - \cosh\beta_{f,r}x)\right]\end{aligned} \tag{139}$$

where $A_{f,r}$ is a normalizing factor, selected such that

$$\int_0^{\bar{x}_f}\hat{m}\Phi_{f,r}^2(x)dx = 1 \tag{140}$$

Thus, $A_{f,r}$ becomes

$$A_{f,r} = \frac{1}{\sqrt{\hat{m}\left[A_{f,r_{P_1}} + A_{f,r_{P_2}} + A_{f,r_{P_3}}\right]}} \tag{141}$$

where

$$A_{f,r_{P_1}} = \frac{\left(\sin\beta_k\bar{x}_f - \sinh\beta_k\bar{x}_f\right)^2}{4\beta_k}\left[-2\cos\beta_k\bar{x}_k\sin\beta_k\bar{x}_k + \sinh 2\beta_k\bar{x}_k - 4\sin\beta_k\bar{x}_k\cosh\beta_k\bar{x}_k + 4\cos\beta_k\bar{x}_k\sinh\beta_k\bar{x}_k\right] \tag{142}$$

$$A_{f,r_{P_2}} = \frac{\left(\sin\beta_k\bar{x}_f - \sinh\beta_k\bar{x}_f\right)\left(\cos\beta_k\bar{x}_f + \cosh\beta_k\bar{x}_f\right)}{\beta_k}\left(\sin\beta_k\bar{x}_f - \sinh\beta_k\bar{x}_f\right)^2 \tag{143}$$

$$A_{f,r_{P_3}} = \frac{\left(\cos\beta_k\bar{x}_f + \cosh\beta_k\bar{x}_f\right)^2}{4\beta_k}\left[2\cos\beta_k\bar{x}_k\sin\beta_k\bar{x}_k + \sinh 2\beta_k\bar{x}_k - 4\cos\beta_k\bar{x}_k\sinh\beta_k\bar{x}_k - 4\sin\beta_k\bar{x}_k\cosh\beta_k\bar{x}_k + 4\beta_k\bar{x}_k\right] \tag{144}$$

B. Forced Response

Let the forcing function in Eq. 129 consist of distributed and concentrated loads so that Eq. 129 can be written as

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \hat{m}\frac{\partial^2 w(x,t)}{\partial t^2} = f(x,t) + F_j(t)\delta(x - x_j) \tag{145}$$

where $\delta(x)$ is the dirac delta function defined as

$$\delta(x - x_j) = \left\{ \begin{array}{l} 1 \;\; if \;\; x = x_j \\ 0 \;\; if \;\; x \neq xj \end{array} \right\} \tag{146}$$

From the expansion theorem, the solution to Eq. 145 is

$$w(x,t) = \sum_{r=1}^{\infty} \Phi_{f,r}(x)\eta_{f,r}(t) \tag{147}$$

where $\eta_{f,r}(t)$ is the generalized modal coordinate, for the forebody beam, that satisfies

$$\ddot{\eta}_{f,r}(t) + 2\zeta_{f,r}\omega_{f,r}\dot{\eta}_{f,r}(t) + \omega_{f,r}^2\eta_{f,r}(t) = N_{f,r}(t) \tag{148}$$

Here, $N_{f,r}(t)$ is a generalized force for the k^{th} mode shape of the forebody beam, defined by[6]

$$N_{f,r}(t) = \int_0^{\bar{x}_f} \Phi_{f,r}(x)f(x,t)dx + \sum_{j=1}^{n} \Phi_{f,r}(x_j)F_j(t) \tag{149}$$

where n is the number of concentrated loads on the beam. Given the loading on the forebody beam, the generalized force for the first mode becomes

$$N_{f,1}(t) = \int_0^{\bar{x}_f} \Phi_{f,1}(x)P_{cf}dx - \int_{\bar{x}_f - L_f}^{\bar{x}_f} \Phi_{f,1}(x)P_{cd}dx \tag{150}$$

Evaluating this expression produces

$$\begin{aligned} N_{f,1}(t) = \frac{A_{f,1}P_{cf}}{\beta_1}\left[\left(-\sin\{\beta_1\bar{x}_f\} + \sinh\{\beta_1\bar{x}_f\}\right)\left(\cos\{\beta_1\bar{x}_f\} + \cosh\{\beta_1\bar{x}_f\} - 2\right)\right] \\ \frac{A_{f,1}P_{cf}}{\beta_1}\left[\left(\cos\{\beta_1\bar{x}_f\} + \cosh\{\beta_1\bar{x}_f\}\right)\left(\sin\{\beta_1\bar{x}_f\} - \sinh\{\beta_1\bar{x}_f\}\right)\right] \\ \frac{A_{f,1}P_{cd}}{\beta_1}\left[\left(-\sin\{\beta_1\bar{x}_f\} + \sinh\{\beta_1\bar{x}_f\}\right)\left(\cos\{\beta_1\bar{x}_f\} - \cos\{\beta_1\left(\bar{x}_f - L_f\right)\} + \cosh\{\beta_1\bar{x}_f\} - \cosh\{\beta_1\left(\bar{x}_f - L_f\right)\}\right)\right] \\ \frac{A_{f,1}P_{cd}}{\beta_1}\left[\left(\cos\{\beta_1\bar{x}_f\} + \cosh\{\beta_1\bar{x}_f\}\right)\left(\sin\{\beta_1\bar{x}_f\} - \sin\{\beta_1\left(\bar{x}_f - L_f\right)\} - \sinh\{\beta_1\bar{x}_f\} + \sinh\{\beta_1\left(\bar{x}_f - L_f\right)\}\right)\right] \end{aligned} \tag{151}$$

The solution to Eq. 148 for the r^{th} mode is[6]

$$\begin{aligned} \eta_{f,r}(t) = \frac{1}{\omega_{f,rd}}\int_0^t N_{f,r}(\tau)\exp^{-\zeta_{f,r}\omega_{f,r}(t-\tau)}\sin\left(\omega_{f,rd}(t-\tau)\right)d\tau + \\ \exp^{-\zeta_{f,r}\omega_{f,r}t}\left[\cos\omega_{f,rd}t + \frac{\zeta_{f,r}}{\left(1 - \zeta_{f,r}^2\right)^{\frac{1}{2}}}\sin\omega_{f,rd}t\right]\eta_{f,r}(0) + \left[\frac{1}{\omega_{f,rd}}\exp^{-\zeta_{f,r}\omega_{f,r}t}\sin\omega_{f,rd}t\right]\dot{\eta}_{f,r}(0) \end{aligned} \tag{152}$$

where $\omega_{f,rd} = \omega_{f,r}\left(1 - \zeta_{f,r}^2\right)^{\frac{1}{2}}$. The initial conditions are given by[6]

$$\begin{aligned} \eta_{f,r}(0) = \int_0^{\bar{x}_f} \hat{m}\Phi_{f,r}(x)w(x,0)dx \\ \dot{\eta}_{f,r}(0) = \int_0^{\bar{x}_f} \hat{m}\Phi_{f,r}(x)\dot{w}(x,0)dx \end{aligned} \tag{153}$$

Evaluating the expressions in Eq. 153, for the first forebody mode, yields

$$ \tag{154} $$

In order to incorporate aeroelastic effects into the model, a few simplifying assumptions are made. First, the vehicle does not stretch or compress along the x-axis. Second, for small displacements, when the beams (vehicle) flexes, there is no change in the x direction displacement. With these assumptions, aeroelastic effects only occur in the z-direction.

The aeroelastic effects can be accounted for by taking the time derivative of Eq. 147,

$$ \dot{w}(x,t) = \sum_{r=1}^{\infty} \Phi_{f,r}(x)\dot{\eta}_{f,r}(t) \tag{155} $$

and including this effect in the expressions for the velocities on the upper and lower surfaces, namely Eqs. 11, 14, and 15. The differential forces on the upper and lower forebody surfaces, taking into account the forward section, becomes

$$ d\mathbf{F}_{cf} = \left(-P_1 - \rho_1 a_1 \left\{\left[u + q\tan\tau_{1,U}\left(x - \bar{x}_f\right)\right]\sin\tau_{1,U} - \left[w - qx + \dot{w}(x,t)\right]\cos\tau_{1,U}\right\}\right) dA_{cf}\mathbf{n}_{cf} \tag{156} $$

$$ d\mathbf{F}_{cd} = \left(-P_2 - \rho_2 a_2 \left\{\left[u - q\tan\tau_{1,L}\left(x - \bar{x}_f\right)\right]\sin\tau_{1,L} + \left[w - qx + \dot{w}(x,t)\right]\cos\tau_{1,L}\right\}\right) dA_{cd}\mathbf{n}_{cd} \tag{157} $$

With these differential forces, stability derivatives due to bending of the vehicle can be determined. For the normal force,

$$ \left(C_Z\right)_{\dot{w}} = \frac{1}{q_\infty S}\left[\int_0^{\bar{x}_f} \left(d\mathbf{F}_{cf}\right)_{z-\dot{w}} + \int_{\bar{x}_f - L_f}^{\bar{x}_f} \left(d\mathbf{F}_{cd}\right)_{z-\dot{w}}\right] = \frac{1}{q_\infty S}\int_0^{\bar{x}_f} -\rho_1 a_1 \dot{w}\cos\tau_{1,U}dx + \int_{\bar{x}_f - L_f}^{\bar{x}_f} -\rho_2 a_2 \dot{w}\cos\tau_{1,L}dx \tag{158} $$

Substituting Eq. 155 into Eq. 158 produces

$$ \begin{aligned} \left(C_Z\right)_{\dot{w}} &= \frac{1}{q_\infty S}\left[\int_0^{\bar{x}_f} \left(d\mathbf{F}_{cf}\right)_{z-\dot{w}} + \int_{\bar{x}_f - L_f}^{\bar{x}_f} \left(d\mathbf{F}_{cd}\right)_{z-\dot{w}}\right] \\ &= \frac{1}{q_\infty S}\int_0^{\bar{x}_f} -\rho_1 a_1 \sum_{r=1}^{\infty}\Phi_{f,r}(x)\dot{\eta}_{f,r}(t)\cos\tau_{1,U}dx + \int_{\bar{x}_f - L_f}^{\bar{x}_f} -\rho_2 a_2 \sum_{r=1}^{\infty}\Phi_{f,r}(x)\dot{\eta}_{f,r}(t)\cos\tau_{1,L}dx \end{aligned} \tag{159} $$

which, for one mode becomes

$$ \begin{aligned} \frac{\partial C_Z}{\partial \dot{\eta}_{f,1}} &= \frac{1}{q_\infty S}\int_0^{\bar{x}_f} -\rho_1 a_1 \Phi_{f,1}(x)\cos\tau_{1,U}dx + \int_{\bar{x}_f - L_f}^{\bar{x}_f} -\rho_2 a_2 \Phi_{f,1}(x)\cos\tau_{1,L}dx \\ &= \frac{1}{q_\infty S}\left[\frac{-2\rho_1 a_1 \cos\tau_{1,U}A_{f,1}}{\beta_1}\left(\sin\beta_1\bar{x}_f - \sinh\beta_1\bar{x}_f\right)\right] \\ &\quad - \frac{\rho_2 a_2 \cos\tau_{1,L}A_{f,1}}{\beta_1 q_\infty S}\left[\left(\sin\beta_1\bar{x}_f - \sinh\beta_1\bar{x}_f\right)\left(\cos\beta_1\left(\bar{x}_f - L_f\right) + \cosh\beta_1\left(\bar{x}_f - L_f\right)\right)\right] \\ &\quad - \frac{\rho_2 a_2 \cos\tau_{1,L}A_{f,1}}{\beta_1 q_\infty S}\left[\left(\cos\beta_1\bar{x}_f + \cosh\beta_1\bar{x}_f\right)\left(-\sin\beta_1\left(\bar{x}_f - L_f\right) + \sinh\beta_1\left(\bar{x}_f - L_f\right)\right)\right] \end{aligned} \tag{160} $$

where $A_{f,1}$ is the normalizing factor associated with the first mode. Using the assumption that the flexible effects only perturb the surface velocities in the z-direction, the axial force stability derivative associated with the flexible effect becomes

$$ \frac{\partial C_X}{\partial \dot{\eta}_{f,1}} = 0 \tag{161} $$

For the pitching moment,

$$ \left(C_M\right)_{\dot{w}} = -\frac{1}{q_\infty S\bar{c}}\left[\int_0^{\bar{x}_f} x\left(d\mathbf{F}_{cf}\right)_{z-\dot{w}} + \int_{\bar{x}_f - L_f}^{\bar{x}_f} x\left(d\mathbf{F}_{cd}\right)_{z-\dot{w}}\right] \tag{162} $$

For the first mode only,

$$
\begin{aligned}
\frac{\partial C_m}{\partial \dot{\eta}_{f,1}} &= \frac{2A_{f,1}\rho_1 a_1 \cos\tau_{1,U}}{\beta_1^2 q_\infty S\bar{c}}\left[1-\cos\beta_1\bar{x}_f+\cos\beta_1\bar{x}_f\cosh\beta_1\bar{x}_f-\cosh\beta_1\bar{x}_f\right] \\
&+\frac{A_{f,1}\rho_2 a_2 \cos\tau_{1,L}}{\beta_1^2 q_\infty S\bar{c}}\left[2-\cos\beta_1 L_f+\beta_1\left(\bar{x}_f-L_f\right)\sin\beta_1 L_f+2\cos\beta_1\bar{x}_f\cosh\beta_1\bar{x}_f\right] \\
&+\frac{A_{f,1}\rho_2 a_2 \cos\tau_{1,L}}{\beta_1^2 q_\infty S\bar{c}}\left[\beta_1\left(\bar{x}_f-L_f\right)\left(\sin\beta_1\bar{x}_f\cosh\beta_1\left(\bar{x}_f-L_f\right)+\cos\beta_1\bar{x}_f\sinh\beta_1\left(\bar{x}_f-L_f\right)\right)\right] \\
&+\frac{A_{f,1}\rho_2 a_2 \cos\tau_{1,L}}{\beta_1^2 q_\infty S\bar{c}}\left[-\sin\beta_1\bar{x}_f\sinh\beta_1\left(\bar{x}_f-L_f\right)-\cos\beta_1\bar{x}_f\cosh\beta_1\left(\bar{x}_f-L_f\right)\right] \\
&+\frac{A_{f,1}\rho_2 a_2 \cos\tau_{1,L}}{\beta_1^2 q_\infty S\bar{c}}\left[\sinh\beta_1\bar{x}_f\sin\beta_1\left(\bar{x}_f-L_f\right)-\cosh\beta_1\bar{x}_f\cos\beta_1\left(\bar{x}_f-L_f\right)\right] \\
&+\frac{A_{f,1}\rho_2 a_2 \cos\tau_{1,L}}{\beta_1^2 q_\infty S\bar{c}}\left[-\beta_1\left(\bar{x}_f-L_f\right)\sinh\beta_1\bar{x}_f\cos\beta_1\left(\bar{x}_f-L_f\right)-\beta_1\left(\bar{x}_f-L_f\right)\cosh\beta_1\bar{x}_f\sin\beta_1\left(\bar{x}_f-L_f\right)\right] \\
&+\frac{A_{f,1}\rho_2 a_2 \cos\tau_{1,L}}{\beta_1^2 q_\infty S\bar{c}}\left[-\beta_1\left(\bar{x}_f-L_f\right)\sinh\beta_1 L_f-\cosh\beta_1 L_f\right]
\end{aligned}
\tag{163}
$$

With the inclusion of the flexible stability derivatives, the total aerodynamic forces and moments on the vehicle are

$$
X_{total}=X_{cf_f}+X_{cf_a}+X_{cd}+X_{gh}+X_{ef}+X_{e_L}+X_{e_U}+q_\infty S\frac{\partial C_X}{\partial\alpha}\alpha+F_{x_{inlet}} \tag{164}
$$

$$
Z_{total}=Z_{cf_f}+Z_{cf_a}+Z_{cd}+Z_{gh}+Z_{ef}+Z_{e_L}+Z_{e_U}+q_\infty S\frac{\partial C_Z}{\partial\alpha}\alpha+q_\infty S\frac{\partial C_Z}{\partial q}\frac{q\bar{c}}{2V_\infty}+F_{z_{inlet}}+q_\infty S\frac{\partial C_Z}{\partial\dot{\eta}_{f,1}}\dot{\eta}_{f,1} \tag{165}
$$

$$
M_{total}=M_{cf_f}+M_{cf_a}+M_{cd}+M_{gh}+M_{ef}+M_{e_L}+M_{e_U}+q_\infty S\bar{c}\frac{\partial C_M}{\partial\alpha}\alpha+q_\infty S\bar{c}\frac{\partial C_M}{\partial q}\frac{q\bar{c}}{2V_\infty}+M_{inlet}+M_{engine}+q_\infty S\bar{c}\frac{\partial C_M}{\partial\dot{\eta}_{f,1}}\dot{\eta}_{f,1} \tag{166}
$$

XII. Equations of Motion for a Flexible Vehicle

The equations of motion for the flexible vehicle are[4]

$$
\begin{aligned}
\dot{V}_T &= \tfrac{1}{m}\left(T\cos\alpha-D\right)-g\sin\left(\theta-\alpha\right) \\
\dot{\alpha} &= \tfrac{1}{mV_T}\left(-T\sin\alpha-L\right)+Q+\tfrac{g}{V_T}\cos\left(\theta-\alpha\right) \\
I_{yy}\dot{Q}-\Psi_f\ddot{\eta}_f &= M \\
k_f\ddot{\eta}_f+2\zeta_f\omega_f\dot{\eta}_f+\omega_f^2\eta_f &= N_f-\Psi_f\tfrac{m}{I_{yy}} \\
\dot{h} &= V_T\sin\left(\theta-\alpha\right)
\end{aligned}
\tag{167}
$$

where

$$
\begin{aligned}
\Psi_f &= \int_0^{\bar{x}_f} x\Phi_{f,1}(x)dx \\
k_f &= 1+\tfrac{\Phi_f}{I_{yy}}
\end{aligned}
\tag{168}
$$

Clearly, the flexible effects are coupled into the pitch rate equation. In addition to this, the bending of the structure has an effect on the angle of attack of the vehicle. Since engine performance is a function of shock angle and shock angle is a function of angle of attack, a significant change to the vehicle's performance can occur due to structural bending. It is assumed that the entire forebody observes the same change in angle of attack as seen at the nose of the vehicle. In other words, the worst case change in angle of attack is used for the entire forebody. This change in angle of attack is computed as:

$$
\Delta\alpha=\arctan\left[\Phi'\left(\bar{x}_f\right)\eta_{f,1}(t)\right] \tag{169}
$$

XIII. Results

At this point, only initial simulation results are available. Currently, the model has been simulated open-loop, with a fixed control surface deflection, to ensure that the model is operating correctly. One point of interest that can be obtained from this simple simulation is the contribution to the forces and moments due to the inclusion of unsteady effects. Figure 6 shows the steady and unsteady X force, Figure 7 shows the steady and unsteady Z force, and Figure 8 shows the steady and unsteady pitching moment. Obviously, the unsteady components will have an impact on the total forces and moments, as these terms are not negligible compared to the steady terms. In fact, the unsteady pitch moment has the largest effect, followed by the unsteady Z force. The unsteady X force has little effect on the total X force.

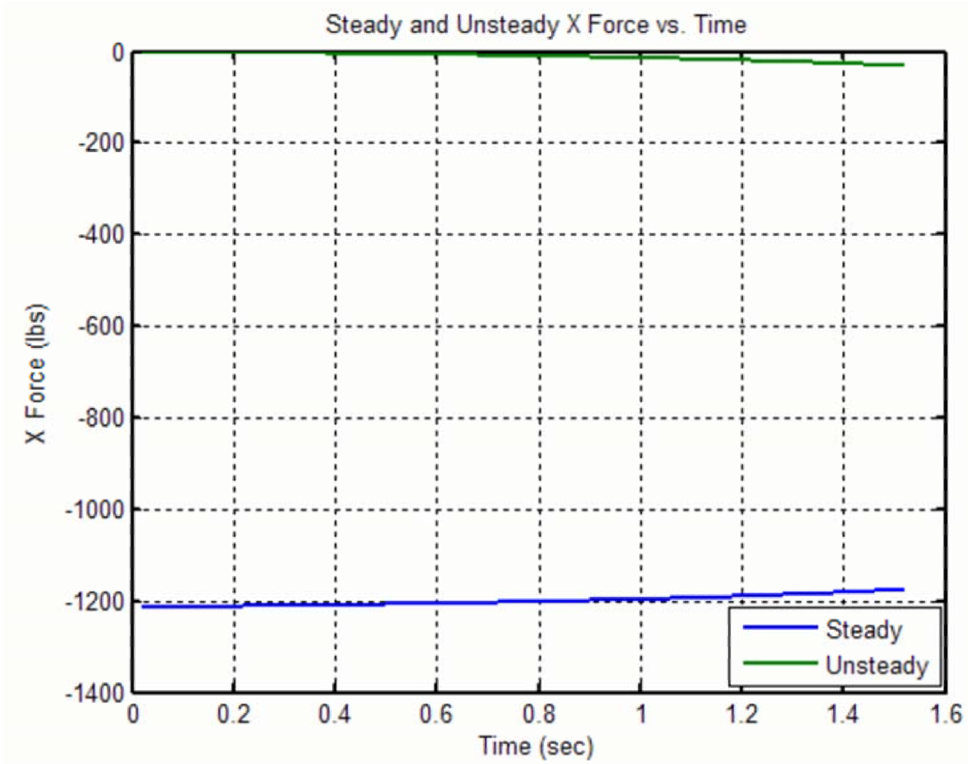

Figure 6. Steady and Unsteady X Forces.

XIV. Conclusions

In this work, piston theory is used to develop a model for the longitudinal dynamics of a hypersonic vehicle. In particular, velocities of flow normal to the surface of the vehicle are used in a first order piston theory framework to determine the pressures on the surfaces of the vehicle. The pressures are then integrated over the body to determine the forces acting on the vehicle. Piston theory is useful here because it allows the inclusion of unsteady aerodynamic effects, which are not captured using other techniques.

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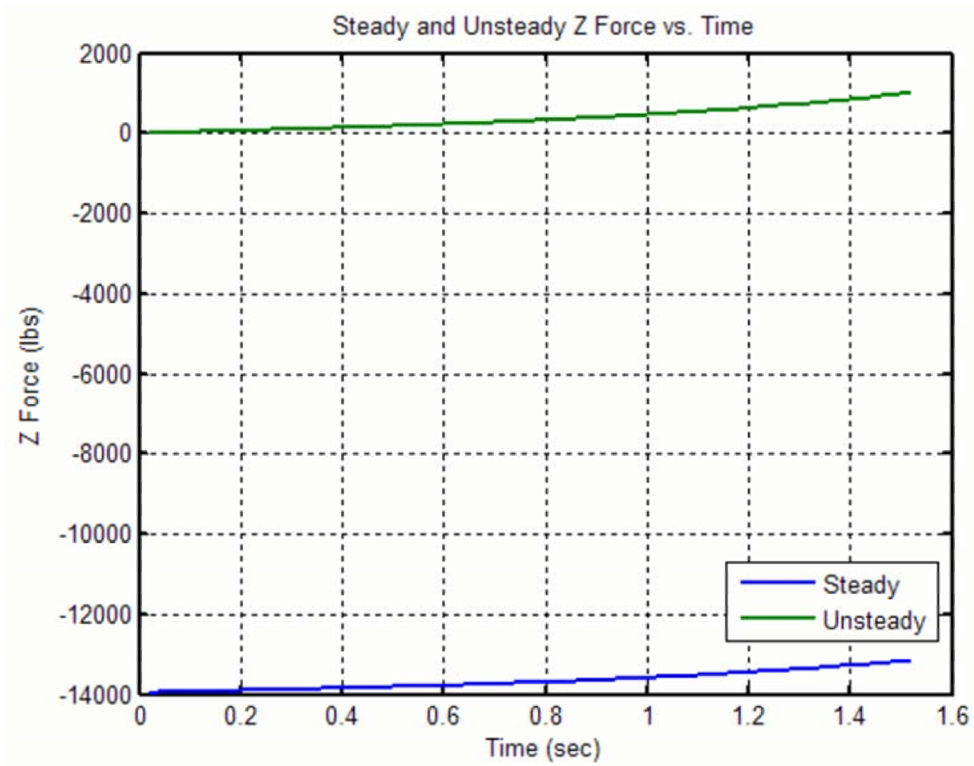

Figure 7. Steady and Unsteady Z Forces.

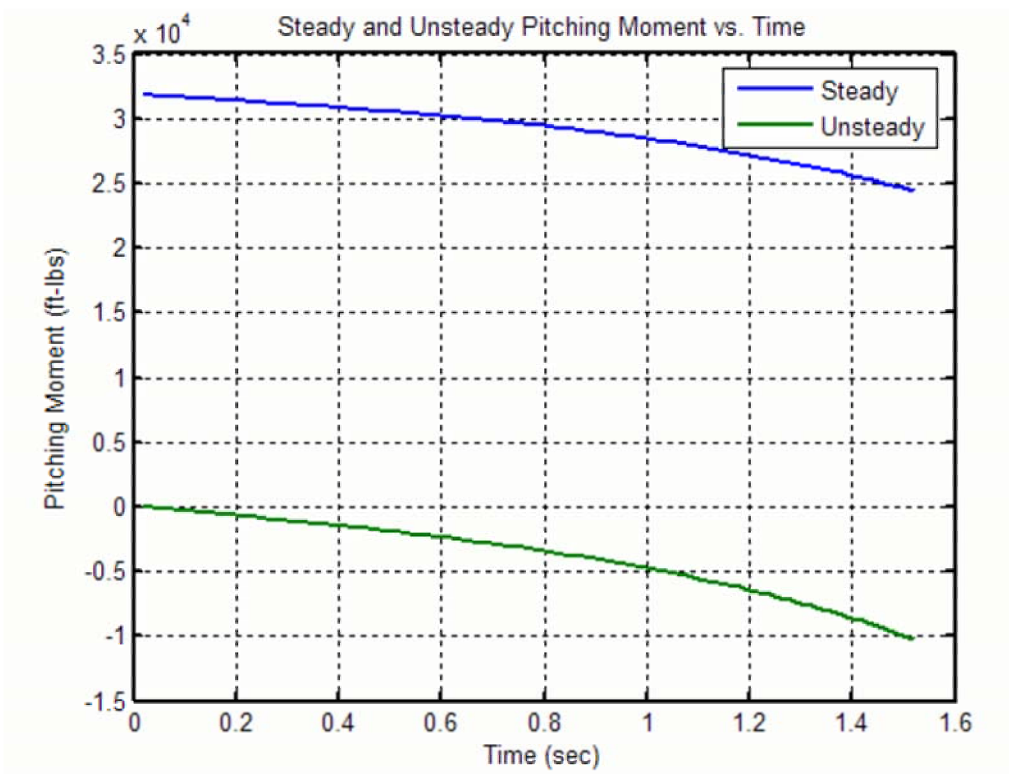

Figure 8. Steady and Unsteady Pitching Moments.